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# REPORT C

# DEFLECTIONS OF A SUPERSONIC WING DUE TO AERODYNAMIC HEATING

by Richard W. Luce Jr.

D.I.C. Project Number 6553

U.S. Air Force Contract Number W33-038 ac-17239

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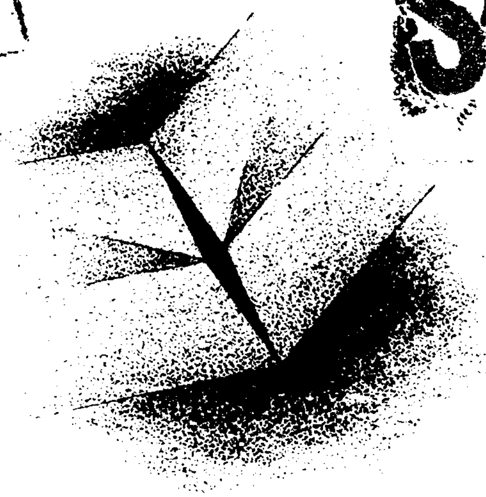

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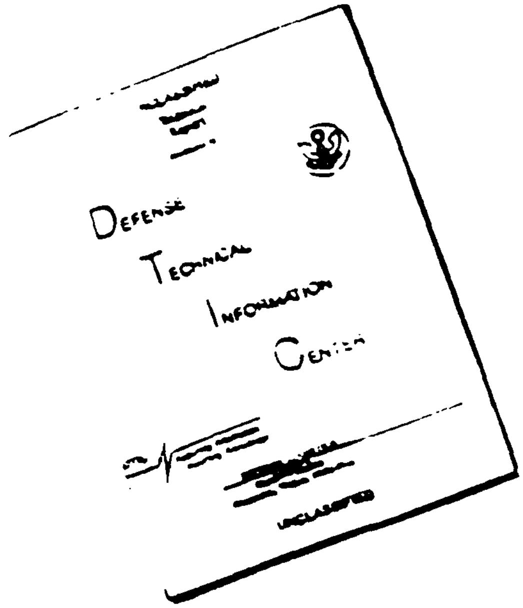


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# DEFLECTIONS OF A SUPERSONIC WING DUE TO AERODYNAMIC HEATING

by

RICHARD W. LUCE, JR.

For the U. S. Air Force under
Contract W33-038 ac-17239
D.I.C. Project No. 6553

June 1, 1949

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# PREFACE

This report is one of a series which is the result of a joint undertaking of the Departments of Aeronautical Engineering, under the supervision of Professors J. S. Newell and H. S. Tsien, and of Mechanical Engineering, under the supervision of Professors J. Kaye and W. M. Rohsenow, at the Massachusetts Institute of Technology for the U. S. Air Forces under Contract No. W33-038 ac-17239. The work was performed in the Division of Industrial Cooperation as projects DIC 6553 and DIC 6580. The work began in October, 1947 and extended to June, 1949.

Since the basic problems relating to the thermal effects in a structure moving at supersonic speeds are relatively new and unsolved, the major efforts in this study were of an exploratory nature. Rather than attempting to cover a large number of different structures, it was deemed advisable to select a particular case, namely that of a thin wedge-shape wing travelling at supersonic speeds, and to discover the basic problems which must be solved in order to design such a structure safely. Even for this particular case only the more important variables could be examined in the time available.

The problem was divided into the following four parts:

a. The determination of the temperature distribution in the wedge-shape wing. The results are given in Report A.

b. The thermal stresses in the wing caused by this non-uniform distribution. The results are given in Report B.

c. The effects of this temperature distribution on the aerodynamic behavior of the wing. The results are given in Report C.

d. A search of the literature in order to discover any previous work in this field. The results are given in Report D.

## SUMMARY

In this report, a study is made of the deflections of a wing due to aerodynamic heating, and the effects of these deflections on the aerodynamic performance of the wing. The temperature distribution used is that calculated in Report A of this series by a method of finite differences. A different approach to the heat transfer problem is suggested in an Appendix to this Report. For the sake of simplicity, the wing which is considered is a flat rectangular plate of constant thickness. However, methods of generalizing the procedure to solid wings of variable thickness are indicated.

The analysis is based upon a strain energy procedure rather than a solution of the differential equations of thermo-elasticity. In this procedure, it was found necessary to take into account the coupling of stresses in the plane of the plate with transverse deflections of the plate.

Results are given in tabular and graphical form, and show that in general the effect of these deflections is not large. The principal aerodynamic effect is an increase in drag coefficient. However all of these calculations are based upon a Mach number of six, and thus at smaller Mach numbers, the deflections will definitely not reach serious proportions.

<u>TABLE OF CONTENTS</u>

# INTRODUCTION

As shown by the results of calculations in the previous reports of this series, non-uniform heating by rapid acceleration to supersonic speeds in dense atmosphere leads to very large thermal stresses in the material. These thermal stresses are calculated on the assumption that the dimensions of the wing are infinitely large. If the dimensions of the wing are finite, there will be some release of stresses due to the effects of free edges. To determine this effect is part of the aim of this report. However, since the material will be assumed to be completely elastic, while plasticity of the material must necessarily appear at the high prevailing temperature, the magnitude of the stresses so calculated cannot be considered as quantitative. The main purpose of the present study is rather the determination of the deformation of the wing due to aerodynamic heating. As will be seen later, the deformation is independent of the value of Young's modulus and is only a function of the temperature distribution and the thermal coefficient of expansion of the material. Therefore, the appearance of plasticity at high temperature and the resultant changes in the _effective_ elastic modulus, will not essentially modify the calculated results on deformation. Hence, the assumption of complete elasticity is justified for the calculation of the deformation of the wing.

Specifically then, this report attempts to obtain a first order estimate of the transverse deflection of a cantilever wing caused by the thermal stresses set up by a transient temperature distribution. The wing is assumed to be accelerated at a constant rate starting at a Mach number of 1.4. The acceleration is assumed to occur at constant altitude and constant wing loading. The angle of attack is thus determined to maintain this constant wing loading during the period of acceleration. During this period of acceleration there is heat transfer from the air into the wing, producing a temperature distribution as a function of time. This temperature distribution has been calculated for a particular set of conditions by a method of finite differences as given in Report A of this series and it is this temperature distribution which will be used in the following work. *

To calculate the deformation and the stresses with a given temperature distribution, one can proceed by either of two ways: 1) To solve the exact equilibrium equations and the equation of compatibility; 2) To use the energy method. This first method is more exact but the labor involved in actual computation is prohibitive for the case of a cantilever wing. The second method will thus be used. This method has the advantage of being flexible and can be adapted without any difficulty to wings with arbitrary thickness variation. For the detailed analysis given below, the thickness of the wing is assumed to be constant to simplify the calculation. This is believed to be justified for the present study, as the aim is to obtain a first order estimate of the deformation rather than the exact numerical value.

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*In an Appendix to the present Report, an alternate method based upon the Laplace Transform will be presented.

## NOTATIONS

1) $C_d$ = local drag coefficient

2) $C_l$ = local lift coefficient

3) $C_m$ = local moment coefficient

4) $C_p = \dfrac{p - p^0}{\frac{1}{2}\rho^0 U^2}$ = pressure coefficient

5) $D = \dfrac{E h^3}{12(1-\nu^2)}$ = plate bending stiffness

6) $E$ = Young's Modulus of Elasticity

7) $G = \dfrac{E}{2(1+\nu)}$

8) $h$ = plate thickness

9) $l$ = semichord

10) $L$ = semispan

11) $\mathcal{L}(\;)$ = Laplace transform of ( )

12) $N_x = h\sigma_x$ = force in x-direction per foot

13) $N_y = h\sigma_y$ = force in y-direction per foot

14) $N_{xy} = h\tau_{xy}$ = shear load per foot

15) $R = L/l$

16) $S = h/L$

17) $T$ = temperature

18) $T_m$ = temperature averaged with respect of Z

19) $u$ = displacement in the x-direction

20) $v$ = displacement in the y-direction

21) $V$ = potential energy

22) $w$ = displacement in the Z direction

23) $W$ = strain energy function

24) $x$ = spanwise coordinate

25) $y$ = chordwise coordinate

26) $z$ = transverse coordinate

27) $\alpha$ = linear thermal coefficient of expansion

28) $\gamma_{xy}$ = shearing strain

29) $\epsilon_x$ = tensile strain in the x-direction

30) $\epsilon_y$ = tensile strain in the y-direction

31) $\eta$ = $y/\ell$

32) $\theta$ = time

33) $\nu$ = Poisson's ratio

34) $\xi$ = $x/L$

35) $\Pi$ = strain energy of a flat plate

36) $\Pi_1$ = strain energy due to bending stresses

37) $\Pi_2$ = strain energy due to plane stresses

38) $\sigma_x$ = tensile stress in the x-direction

39) $\sigma_y$ = tensile stress in the y-direction

40) $\tau_{xy}$ = shearing stress

41) $\tau$ = thickness ratio = $h/2\ell$

42) $\phi$ = Airy's stress function

## FORMULATION OF THE PROBLEM

The coordinate system and dimensions to be used in the analysis are as shown in Figure 1. The following set of non-dimensional parameters will also be used in the analysis.

$$R = L/\ell, \qquad S = h/L$$
$$\xi = x/L, \qquad \eta = y/\ell \tag{1}$$

The results of the finite difference procedure indicate* that the temperature distribution at any instant can be approximated quite closely by an expression of the following form, where $T(x,y,z)$ is the local temperature at the point $(x,y,z)$ above the uniform temperature before heating.

$$T(x,y,z) = A(y) + B(y)\frac{z}{h} + C(y)\left(\frac{z}{h}\right)^2 \tag{2}$$

*See Report C

The coefficients A, B, and C are only functions of the span-wise coordinate y. The influence of the wing tips on the temperature distribution is thus neglected.

From this expression, it is easily seen that A, B, and C have the following significance.

$$A(y) = T_{z=0}$$

$$B(y) = T_{z=h/2} - T_{z=-h/2} \tag{3}$$

$$C(y) = 2\left[T_{z=h/2} + T_{z=-h/2}\right] - 4T_{z=0}$$

It is also found convenient to define the mean temperature as:

$$T_m = \frac{1}{h}\int_{-h/2}^{h/2} T\,dZ = A(y) + C(y)\frac{h}{12} \tag{4}$$

The deflections of a wing are mainly due to the bending of the wing surface. But bending is the result of the difference in expansion or the difference in temperature of the material at the top and the bottom surfaces. If this is true, then only the linear term $B(y)\,z/h$ in the temperature distribution will influence the deflections. However, this is not true: Due to the non-uniformity of the mean temperature defined in (4), there is rather large stretching and compression of the median surface, lying half-way between the top and the bottom surfaces. The stretching and compression of the median surface yield an average tensile or compressive stress in the wing. These are the stresses in the plane of the wing. Such stresses will influence the bending of the wing through coupling effects. Therefore, to calculate the deflection of the wing, the procedure can be divided into two steps:

1) Calculate the average or plate stresses
2) Calculate the bending with the coupling effects of the plate stresses.

These steps will be treated separately in the following sections.

## PLATE STRESSES

Let

$\varepsilon_x$ = tensile strain in the x-direction

$\varepsilon_y$ = tensile strain in the y-direction

$\gamma_{xy}$ = shearing strain

$\sigma_x$ = tensile stress in the x-direction

$\sigma_y$ = tensile stress in the y-direction

$\tau_{xy}$ = shearing stress

$E$ = Young's modulus

$\nu$ = Poisson's ratio

$\alpha$ = linear thermal expansion coefficient.

Then the relations between the strain and stress are:

$$
\begin{aligned}
\varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha T_m \\
\varepsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha T_m \qquad (5) \\
\gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy}
\end{aligned}
$$

The equilibrium equations are:

$$
\begin{aligned}
\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} &= 0 \\
\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} &= 0
\end{aligned} \qquad (6)
$$

In addition, the compatibility equations become:

$$
\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\,\partial y} \qquad (7)
$$

Combining the stress-strain equations and the compatibility equation:

$$
\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) + E\alpha\nabla^2 T_m = 2(1+\nu)\frac{\partial^2\tau_{xy}}{\partial x\,\partial y} \qquad (8)
$$

But differentiating the equilibrium equations with respect to x and y respectively, and adding:

$$
2\frac{\partial^2\tau_{xy}}{\partial x\,\partial y} = -\left(\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\sigma_y}{\partial y^2}\right) \qquad (9)
$$

Hence by eliminating $\tau_{xy}$ between (8) and (9):

$$\frac{\partial^2\sigma_x}{\partial y^2} - \nu\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\sigma_y}{\partial x^2} - \nu\frac{\partial^2\sigma_y}{\partial y^2} + E\alpha\nabla^2 T_m$$

$$= (1+\nu)\left(-\frac{\partial^2\sigma_x}{\partial x^2} - \frac{\partial^2\sigma_y}{\partial y^2}\right) \qquad (10)$$

or: $$\nabla^2(\sigma_x + \sigma_y) = -E\alpha\nabla^2 T_m \qquad (11)$$

Now introduce the stress function $\phi$ defined by:

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y} \qquad (12)$$

(11) becomes:

$$\nabla^2\nabla^2\phi = -E\alpha\nabla^2 T_m \qquad (13)$$

Thus the differential equation (13) must be solved subject to the boundary conditions of zero normal and shear stress on the free edges. The supported end of the plate is assumed to be free to displace in the y-direction. Tha same boundary conditions are satisfied if the cantilever plate is considered as one half of a rectangular plate free at all edges. Thus for convenience, the cantilever plate is replaced by an equivalent free plate of twice the length, as shown in Figure 2.

The boundary conditions become:

$$\left.\begin{array}{ll} \text{At } y = \pm\ell, & \phi_{xy} = \phi_{xx} = 0 \\ \text{At } x = \pm L, & \phi_{xy} = \phi_{yy} = 0 \end{array}\right\} \qquad (14)$$

A particular solution to equation (1) is:

$$\phi_p = -E\alpha\iint T_m\,dy\,dy \qquad (15)$$

Then: $$\phi_{p\,xy} = \phi_{p\,xx} \equiv 0$$

$$\phi_{p\,yy} = -E\alpha T_m$$

The problem is thus reduced to that of finding a homogeneous solution to the equation:

$$\nabla^2 \nabla^2 \phi_H = 0 \tag{16}$$

With the boundary conditions:

$$\begin{aligned} \text{At} \quad x = \pm L, \quad & \phi_{Hxy} = 0 \\ & \phi_{Hyy} = -\phi_{Pyy} = E\alpha T_m \end{aligned} \tag{17}$$

$$\text{At} \quad y = \pm \ell, \quad \phi_{Hxy} = \phi_{Hxx} = 0$$

But this system is identical with that of finding the stress distribution in a plate loaded as shown in Figure 3 .

SOLUTION OF THE PLANE STRESS PROBLEM

This problem can be solved by a strain energy procedure developed for plane stress. The problem is equivalent to that of finding a function $\phi_H$ which satisfies the boundary conditions and minimizes the integral: *

$$V = \frac{1}{2E} \iint \left\{ (\phi_{Hyy})^2 + (\phi_{Hxx})^2 + 2(\phi_{Hxy})^2 \right\} dx\, dy \tag{18}$$

A solution is assumed in the form,

$$\phi_H = E\alpha \iint T_m \, dy\, dy + (\xi^2 - 1)^2 (\eta^2 - 1)^2 \left[ \alpha_1 + \alpha_2 \xi^2 + \alpha_3 \eta \right] \tag{19}$$

where the $\alpha_i$ are undetermined coefficients.

It can easily be seen that this function automatically satisfies the boundary conditions, and hence it is only necessary to minimize the strain energy. It is assumed that the mean temperature can be expressed in a power series as given below:

$$T_m = \sum_{n=0}^{\infty} t_n \eta^n$$

*See for instance: Timoshenko, "Theory of Elasticity", McGraw-Hill, pp. 151.

Substituting these expressions into the expression for strain energy, there follows:

$$\frac{L^3}{2}EV = L^4\iint (E\alpha T_m)^2 \, d\xi \, d\eta + \sum_{n=0}^{\infty} 2E\alpha\frac{L^4}{2^2} t_n\Big[\frac{8}{15}\alpha_1(1+(-1)^n)\left(\frac{12}{n+3} - \frac{4}{n+1}\right)$$

$$+ \frac{8}{105}\alpha_2(1+(-1)^n)\left(\frac{12}{n+3} - \frac{4}{n+1}\right) + \frac{8}{15}\alpha_3(1-(-1)^n)\left(\frac{20}{n+4} - \frac{12}{n+3}\right)\Big]$$

$$+ [10.4R^4 + 5.93R^2 + 10.42]\alpha_1^2 + [.217R^4 + .501R^2$$

$$+ 4.48]\alpha_2^2 + [5.81R^4 + 1.962R^2 + .396]\alpha_3^2$$

$$+ [1.892R^4 + 1.49]\alpha_1\alpha_2 \qquad (21)$$

where R is the "aspect ratio"

But the minimizing conditions are that:

$$\frac{\partial\left(\frac{L^3}{2}EV\right)}{\partial\alpha_i} = 0$$

Or,

$$\frac{\alpha_3}{E\alpha L^2} = \frac{-R^2}{10.8R^4 + 3.68R^2 + 1.867}\sum_{n=0}^{\infty}(1-(-1)^n)\left(\frac{20}{n+4} - \frac{12}{n+3}\right) t_n$$

$$[19.5R^4 + 11.11R^2 + 19.55]\frac{\alpha_1}{E\alpha L^2} + [1.773R^4 + 1.396]\frac{\alpha_2}{E\alpha L^2}$$

$$= -R^2\sum_{n=0}^{\infty}(1+(-1)^n)\left(\frac{12}{n+3} - \frac{4}{n+1}\right) t_n$$

$$[12.42R^4 + 9.775]\frac{\alpha_1}{E\alpha L^2} + [2.845R^4 + 6.57R^2 + 58.8]\frac{\alpha_2}{E\alpha L^2}$$

$$= -R^2\sum_{n=0}^{\infty}(1+(-1)^n)\left(\frac{12}{n+3} - \frac{4}{n+1}\right) t_n \qquad (22)$$

<u>As an example</u>, the wing whose temperature distribution was calculated before by the finite difference method,* will be analyzed. This wing has a thickness h of 0.22 ft. and a semi-chord $\ell$ of 2.75 ft. The thickness ratio is thus 8%. The wing is loaded to 100 lbs. per square foot and is accelerated at 50,000 ft. altitude from Mach number 1.4 to Mach number 6 at 1 Mach number every 30 seconds. The calculated temperature distribution at Mach number 6 is given in Report A. Expressing this temperature distribution as a polynomial as expressed by (20), the following coefficients are determined:

$$
\begin{aligned}
t_0 &= 317 \\
t_1 &= -184.2 \\
t_2 &= 252 \\
t_3 &= 765 \\
t_4 &= 2135 \\
t_5 &= 2638 \\
t_6 &= 0 \\
t_7 &= -2057
\end{aligned}
\qquad (23)
$$

Using a value of R = 3, the three equations become:

$$\frac{\alpha_3}{E\alpha L^2} = -3.22$$

$$1813\frac{\alpha_1}{E\alpha L^2} + 154.9\frac{\alpha_2}{E\alpha L^2} = -45,500$$

$$154.9\frac{\alpha_1}{E\alpha L^2} + 53.12\frac{\alpha_2}{E\alpha L^2} = -6500$$

whose solutions are:

$$
\begin{aligned}
\frac{\alpha_3}{E\alpha L^2} &= -3.22 \\
\frac{\alpha_1}{E\alpha L^2} &= -19.5 \\
\frac{\alpha_2}{E\alpha L^2} &= -65.4
\end{aligned}
\qquad (24)
$$

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*In an Appendix to the present Report, an alternate method based upon the Laplace transform will be presented.

Using those values, the stress function becomes:

$$\phi = \phi_H + \phi_P$$

$$= E\alpha L^2(\xi^2-1)^2(\eta^2-1)^2\left[-19.5 - 65.4\xi^2 - 3.22\eta\right] \qquad (25)$$

The stresses then are,

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = \frac{1}{\ell^2}\frac{\partial^2\phi}{\partial\eta^2}$$

$$= E\alpha R^2\Big\{8\eta(\eta^2-1)(\xi^2-1)^2(-3.22) + (12\eta^2-4)(\xi^2-1)^2(-19.5-65.4\xi^2-3.22\eta)\Big\} \qquad (26)$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2}$$

$$= E\alpha\Big\{(\xi^2-1)^2(\eta^2-1)^2(-130.8) + 8\xi(\xi^2-1)(\eta^2-1)^2(-130.8\xi) + (12\xi^2-4)(\eta^2-1)^2(-19.5-65.4\xi^2-3.22\eta)\Big\} \qquad (27)$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y}$$

$$= -E\alpha R\Big\{-3.22(4\xi)(\xi^2-1)(\eta^2-1)^2 + 4\eta(\eta^2-1)(\xi^2-1)^2(-130.8\xi) + 8\eta\xi(\xi^2-1)(\eta^2-1)(-19.5-65.4\xi^2-3.22\eta)\Big\} \qquad (28)$$

The results of the calculation are presented in Fig. 5 . It is seen that only the direct stress in the span-wise direction or $\sigma_x$ is large. This is to be expected as the large temperature changes in the chord-wise direction would require large changes in the span-wise expansion along a chord. But such large variations are prevented by the continuity of the material in the span-wise direction. This explains large values for the span-wise direct stress. Near the leading edge and the trailing edge, where the temperature is high, the material tends to expand more than allowed, thus $\sigma_x$ is negative or compressive. Near the mid-chord, the temperature is low; thus the material tends to expand less than allowed, the $\sigma_x$ is positive or the stress is a tension. Of course, at the wing tip, $\xi = 1$, the direct stress $\sigma_x$ has to be zero; therefore $\sigma_x$ must decrease as $\xi$ increases from $\xi = 0$ to $\xi = 1$. This decrease is however rather slow for small values of $\xi$. Hence, the tip effect is limited to a rather small region.

It should be noted that the above result is obtained without any assumption as to the values fo E and $\alpha$. In fact the stresses are directly proportional to the product $E\alpha$. The only other explicit parameter is the aspect ratio R of the wing. However, one must bear in mind the effect of scale on the heat transfer and thus the calculated result is not strictly applicable to wings of other dimensions. The influence of R can be easily calculated by going back to (22).

## BENDING DEFLECTION

Having calculated the plane stress distribution, it is now desired to introduce it into the calculation of the transverse deflections of the plate. To do this, the total strain energy stored in the plate as it deflects is divided into two parts; first that due to the components of thermal stress producing pure bending moments; and second that due to the coupling of transverse deflections and plane stress. It is the sum of these two energy components which is then to be minimized. First the energy due to the bending stresses will be developed.

To calculate this expression, the usual plate assumptions on the stresses are made. These assumptions are that:

$$\sigma_z, \ \tau_{xz}, \ \tau_{yz} \ll \sigma_x, \ \sigma_y, \ \tau_{xy}$$

Accordingly, the equilibrium equations and stress-strain equations become:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \qquad (29)$$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu \sigma_y) + \alpha T$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu \sigma_x) + \alpha T$$

$$\varepsilon_z = \frac{-\nu}{E}(\sigma_x + \sigma_y) + \alpha T$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad (30)$$

$$\gamma_{xz} = 0$$

$$\gamma_{yz} = 0$$

where $G = \dfrac{E}{2(1+\nu)}$

The first, second and fourth of the stress-strain equations can be rewritten as:

$$\sigma_x = \frac{E}{1-\nu^2}\left[\varepsilon_x + \nu\varepsilon_y - (1+\nu)\alpha T\right]$$

$$\sigma_y = \frac{E}{1-\nu^2}\left[\varepsilon_y + \nu\varepsilon_x - (1+\nu)\alpha T\right] \qquad (32)$$

$$\tau_{xy} = G\gamma_{xy}$$

The next assumption is that the deflection $w$ may be approximated by $w(x, y)$ , the deflection of the mid plane of the plate. With this assumption, the last two stress strain equations can be integrated directly to yield:

$$u = -z\frac{\partial w}{\partial x}$$

$$v = -z\frac{\partial w}{\partial y} \qquad (33)$$

It can be seen that these expressions are equivalent to the usual bending assumptions of displacement proportional to distance from the neutral axis.

The strain energy can be expressed as an integral:

$$\Pi = \iiint W \, dx \, dy \, dz \tag{34}$$

where W is a strain energy function such that:

$$\sigma_x = \frac{\partial W}{\partial \varepsilon_x}$$

$$\sigma_y = \frac{\partial W}{\partial \varepsilon_y}$$

$$\tau_{xy} = \frac{\partial W}{\partial \gamma_{xy}}$$

Integrating the expressions and substituting for the stresses from the stress strain equations:

$$W = \frac{E}{2(1-\nu^2)}\left\{(\varepsilon_x + \varepsilon_y)\left[(\varepsilon_x + \varepsilon_y) - 2(1+\nu)\alpha T\right] + 2(1-\nu)\left(\frac{\gamma_{xy}^2}{4} - \varepsilon_x \varepsilon_y\right)\right\} \tag{35}$$

But:

$$\varepsilon_x = -Z \frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = -Z \frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = -2Z \frac{\partial^2 w}{\partial x \partial y}$$

It is only the terms involving $w$ in these expressions that effect the bending strains, and hence the energy function W for bending becomes:

$$W = \frac{E}{2(1-\nu^2)}\left\{Z^2\left[(\nabla^2 w)^2 + \frac{2(1+\nu)\alpha T \, \nabla^2 w}{Z}\right.\right.$$

$$\left.\left. + 2(1-\nu)\left(\left[\frac{\partial^2 w}{\partial x \partial y}\right]^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)\right]\right\} \tag{37}$$

Thus integrating this function directly with respect to Z:

$$\Pi_1 = \frac{D}{2}\iint\left\{(\nabla^2 w)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] + 2(1+\nu)\kappa \frac{B}{h}\nabla^2 w\right\} dx\,dy \qquad (38)$$

where B is the coefficient of the linear term in (2).

The expression for the strain energy due to the coupling of plane stress and transverse deflections is:*

$$\Pi_2 = \frac{1}{2}\iint\left[N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] dx\,dy \qquad (39)$$

Where:

$$N_x = h\sigma_x$$

$$N_y = h\sigma_y$$

$$N_{xy} = h\tau_{xy}$$

The total energy of the plate is thus the sum of these two components, namely:

$$\Pi = \Pi_1 + \Pi_2$$

The equilibrium position of the plate is determined by minimizing the energy $\Pi$.

---

*See Timoshenko in his "Theory of Plates and Shells", pp 305.

## SOLUTION FOR THE BENDING DEFLECTION

It is now necessary to assume a solution for the displacement which satisfies the necessary boundary conditions on the edges where displacements are prescribed, and which contains undetermined coefficients to be determined by the variational procedure. In the case of the cantilever plate, the only edge on which displacements are prescribed is the clamped root, and hence the free edge boundary conditions do not need to be explicitly satisfied in the form of the assumed deflection. This fact, which can be substantiated in Trefftz for example*, does not seem to be generally mentioned in well-known texts on the theory of elasticity. Physically this is easy to understand, as a free edge would certainly seek the position of minimum strain energy among all kinematically possible forms of deflection. This situation is of fundamental importance, because it is the satisfying of the free edge boundary conditions which makes the solution of the cantilever plate problem difficult to obtain by other methods. A form of the deflection which meets these requirements, and which will be used in the variational procedure, is as follows:

$$w = \left[a_1\xi^2 + b_1\xi^3\right] + \left[a_2\xi^2 + b_2\xi^3\right]\eta + \left[a_3\xi^2 + b_3\xi^3\right]\eta^2 \qquad (42)$$

Then the second group of terms represents the change in angle of attack of the section and the third group of terms represents the change in the curvature of the section. Introducing this expression, and the previously calculated $\sigma_x$, $\sigma_y$, AND $\tau_{xy}$ into the strain energy expression, and assuming that:

$$B/h = \sum_{k=0}^{\infty} \alpha_k \eta^k$$

where the $\alpha_k$ have to be determined from the given temperature distribution:

$$\frac{R\Pi}{Eh} = \left\langle \frac{\xi^2}{24(1-\nu^2)} \left\{ \left[ 8a_1^2 + 24a_1b_1 + 24b_1^2 + \frac{16}{3}a_1a_3 + 8b_1a_3 \right.\right.\right.$$

$$+ 8b_3a_1 + 16b_1b_3 + 8/3\, a_2^2 + 8a_2b_2 + 8b_2^2 + 8/5\, a_3^2$$

$$\left. + \frac{24}{5}a_3b_3 + \frac{24}{5}b_3^2 \right] + R^2\left[\frac{16}{3}a_1a_3 + 12b_1a_3 + 4a_1b_3\right.$$

*Trefftz: Mathematische Elastizitatstheorie. Handbuch der Physik, Vol. 6, Kap. 2, pp. 66, (1928)

$$+\tfrac{48}{5}b_1b_3+\tfrac{16}{5}a_3^2+\tfrac{16}{5}a_3b_3+\tfrac{48}{5}b_3^2\Big]+R^3\Big[\tfrac{8}{5}a_3^2$$

$$+\tfrac{8}{3}a_3b_3+\tfrac{8}{7}b_3^2\Big]+2(1-\nu)R^2\Big[\tfrac{8}{3}a_2^2+6a_2b_2+\tfrac{18}{5}b_2^2$$

$$+\tfrac{24}{5}a_3^2+\tfrac{16}{5}a_3b_3+\tfrac{16}{5}b_3^2-\tfrac{8}{3}a_1a_3-6b_1a_3-2a_1b_3$$

$$-\tfrac{24}{5}b_1b_3\Big]+2(1+\nu)\alpha\sum_{k=0}^{\infty}L^2\alpha_k\Big[\frac{1+(-1)^k}{k+1}(2a_1+3b_1)$$

$$+\frac{1-(-1)^k}{k+2}(2a_2+3b_2)+\frac{2}{3}\,\frac{1+(-1)^k}{(k+1)(k+3)}(3k+3+3R^2+R^2k)a_3$$

$$+\frac{1+(-1)^k}{2(k+1)(k+3)}(6k+3+3R^2+R^2k)b_3\Big]\Big\}$$

$$+\frac{\alpha R^2}{2}\Big\{a_1^2(1.22A_1+.406A_2)+a_1b_1(2A_1+0.8A_2)$$

$$+b_1^2(.914A_1+.415A_2)+a_1a_3(1.021A_1-.2165A_2)$$

$$+a_1b_3(-1.067A_1-.1422A_2)+a_3b_1(-1.067A_1-.427A_2)$$

$$+b_1b_3(.743A_1-.2955A_2)+a_2^2(1.30A_1+.2165A_2)$$

$$+a_2b_2(1.067A_1+.427A_2)+b_2^2(1.184A_1+.271A_2)$$

$$+a_3^2(1.579A_1-.031A_2)+a_3b_3(-1.626A_1-.1626A_2)$$

$$+b_3^2(1.208A_1+.163A_2)+a_3b_2(-.031A_3)$$

$$+b_2b_3(-.033A_3)\Big\}\Big\rangle \qquad (44)$$

where $A_1 = \frac{\alpha l}{E\alpha L^2}$ , $S = h/L$

from the plane stress calculations .

As a concrete example, the wing considered in the Section on plane stress will be used again. From previous temperature calculations, the coefficients in the expansion for $\Theta/h$ are as follows:

$\alpha_0 = -240$ $\qquad$ $\alpha_4 = -785$

$\alpha_1 = -51.6$ $\qquad$ $\alpha_5 = 355.6$

$\alpha_2 = -554$ $\qquad$ $\alpha_6 = 1569$

$\alpha_3 = -304$

Assuming the wing is made of steel, introducing a value of $\alpha = 6.28 \times 10^{-6}$, $\nu = .3$, and a value of R = 3, the equation becomes,

$$\frac{2\pi}{3E\alpha h} = \Big[1620\Big\{8a_1^2S^2 + 24a_1b_1S^2 + 24b_1^2S^2 + 19.73a_1a_3S^2$$

$$+40.4\,b_1a_3S^2 + 18.8\,b_3a_1S^2 + 42\,b_1b_3S^2$$

$$+36.3\,a_2^2S^2 + 83.6\,a_2b_2S^2 + 53.3\,b_2^2S^2$$

$$+94.4\,a_3^2S^2 + 191.9\,a_3b_3S^2 + 162.4\,b_3^2S^2$$

$$-.00255h(2a_1 + 3b_1) - .000196h(2a_2 + 3b_2)$$

$$-.01712\,ha_3 - .01287\,hb_3\Big\}$$

$$+\Big\{-50.3a_1^2 - 91.32a_1b_1 - 45.02b_1^2 - 5.79a_1a_3$$

$$+30.1\,a_1b_3 + 48.70\,a_3b_1 + 4.80\,b_1b_3$$

$$- 39.52\, a_2^2 - 48.70\, a_2 b_2 - 40.76\, b_2^2$$

$$- 28.78\, a_3^2 + 42.92\, a_3 b_3 - 34.17\, b_3^2$$

$$\left. \left. + 1.775\, a_3 b_2 + 2.155\, b_2 b_3 \right\} \right] \tag{46}$$

Combining terms, this becomes,

$$\frac{2\pi}{3E\alpha h} = (12{,}980\, S^2 - 50.3)\, a_1^2 + (38{,}900\, S^2 - 91.32)\, a_1 b_1$$

$$+ (38{,}900\, S^2 - 45.02)\, b_1^2 + (31{,}950\, S^2 - 5.79)\, a_1 a_3$$

$$+ (65{,}500\, S^2 + 48.70)\, b_1 a_3 + (30{,}430\, S^2 + 30.1)\, b_3 a_1$$

$$+ (68{,}000\, S^2 + 4.80)\, b_1 b_3 + (58{,}750\, S^2 - 39.52)\, a_2^2$$

$$+ (135{,}500\, S^2 - 48.70)\, a_2 b_2 + (86{,}300\, S^2 - 40.76)\, b_2^2$$

$$+ (153{,}300\, S^2 - 28.78)\, a_3^2 + (310{,}500\, S^2 + 42.92)\, a_3 b_3$$

$$+ (263{,}000\, S^2 - 34.17)\, b_3^2 + (1.775)\, a_3 b_2$$

$$+ (2.155)\, b_2 b_3 - 4.127\, h\, (2 a_1 + 3 b_1)$$

$$- .3175\, h\, (2 a_2 + 3 b_2) - 27.9\, h\, a_3 - 21.0\, h\, b_3$$

$$(47)$$

Differentiating this expression with respect to the coefficients gives the six equations to determine the $a_i$, $b_i$, AND $c_i$ :

$$(25,960\,S^2 - 100.6)\,a_1 + (38,900\,S^2 - 91.32)\,b_1 + (31,950\,S^2 - 5.79)\,a_3$$

$$+ (30,430\,S^2 + 30.1)\,b_3 = 8.254\,h$$

$$(38,900\,S^2 - 91.32)\,a_1 + (77,800\,S^2 - 45.02)\,b_1 + (65,500\,S^2 + 48.7)\,a_3$$

$$+ (68,000\,S^2 + 4.80)\,b_3 = 12.38\,h$$

$$(117,500\,S^2 - 79.04)\,a_2 + (135,500\,S^2 - 48.7)\,b_2 = .635\,h$$

$$(135,500\,S^2 - 48.7)\,a_2 + (172,600\,S^2 - 81.52)\,b_2 + 1.775\,a_3 \qquad (48)$$

$$+ 2.155\,b_3 = .9525\,h$$

$$(31,950\,S^2 - 5.79)\,a_1 + (65,500\,S^2 + 48.7)\,b_1 + 1.775\,b_2$$

$$+ (306,000\,S^2 - 57.56)\,a_3 + (310,500\,S^2 + 42.92)\,b_3 = 27.9\,h$$

$$(30,430\,S^2 + 30.1)\,a_1 + (68,000\,S^2 + 4.80)\,b_1 + 2.155\,b_2$$

$$+ (310,500\,S^2 + 42.92)\,a_3 + (526,000\,S^2 - 68.34)\,b_3 = 21.0\,h$$

As yet, no assumptions on the geometry of the plate have been made other than R = 3. The expressions for the temperature distribution, however, were derived for a particular chord and thickness ratio, and presumably would be different for other sections. The values used in the temperature calculations were:

$$h = 0.22 \text{ FT.}$$

$$\ell = 2.75 \text{ FT.}$$

Since R has been specified as 3, then L = 8.25 ft, and hence S = .0267. For these values, the coefficients satisfying the six equations are:

$$\begin{aligned} a_1 &= -.04053 \\ b_1 &= .0328 \\ a_2 &= .00118 \\ b_2 &= .002738 \\ a_3 &= .00885 \\ b_3 &= .00853 \end{aligned} \tag{49}$$

Hence, the deflection $w$ in feet becomes:

$$w = \left[-.04053\xi^2 + .0328\xi^3\right] + \left[.00118\xi^2 + .002738\xi^3\right]\eta$$

$$+ \left[.00885\xi^2 + .00853\xi^3\right]\eta^2 \tag{50}$$

A plot of this calculated deflection is given in Figure 6. It is seen that the deflection is greatest at the wing tip. However, even at the wing tip the deflection is quite small. The significance of these results will be discussed in a later Section.

## METHOD OF CALCULATION FOR SOLID WINGS OF VARIABLE THICKNESS

The procedure developed in this work is applicable to wings of variable thickness provided that several modifications are made. The only changes involve different limits of integration in the expressions for the various strain energies. In all cases, the thickness $h(x,y)$ must be left inside the integral when integrating with respect to x, and y. The modifications are as follows:

Equation (18) becomes:

$$V = \frac{1}{2E}\iint h(x,y)\left\{\phi_{xxx}^2 + \phi_{xyy}^2 - 2\,\phi_{xxy}^2\right\}dx\,dy \qquad (18a)$$

Equation (38) becomes:

$$\Pi_1 = \frac{1}{2}\frac{E}{1-\nu^2}\iint\left\{\frac{[h(x,y)]^2}{12}\left[(\nabla^2 w)^2 + 2(1+\nu)\,\alpha\,\frac{B\,\nabla^2 w}{h(x,y)} + 2(1-\nu)\left(w_{xy}^2 - w_{xx}w_{yy}\right)\right]\right\}dx\,dy \qquad (38a)$$

And equation (39) becomes:

$$\Pi_2 = \frac{1}{2}\iint h(x,y)\left\{\sigma_x\left(\frac{\partial w}{\partial x}\right)^2 + \sigma_y\left(\frac{\partial w}{\partial y}\right)^2 + 2\tau_{xy}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right\}dx\,dy \qquad (39a)$$

Using these modified expressions, the rest of the procedure may be carried out exactly as in the case of the constant thickness wing.

## GENERAL CONCLUSIONS ON THE DEFLECTION BY AERODYNAMIC HEATING

For a wing of thickness equal to 0.22 ft., chord 2.75 ft., accelerated at an altitude of 50,000 ft., 100 lbs per ft.$^2$ loading and 1 Mach number every 30 seconds, the deflections due to aerodynamic heating at Mach number 6 are given in Table 1 for various values of the coefficient of linear thermal expansions and aspect ratios. The method used in making these calculations is explained in the previous Sections. This Table also gives the changes in the aerodynamic coefficients of the wing due to such deflections. The aerodynamic coefficients are calculated by using the simple Ackeret formulae* for thin wings. It is seen that in all cases the change in the aerodynamic coefficient are small.

Therefore, the results of the present analysis indicate that the deflection and the effects of deflections due to aerodynamic heating on the aerodynamic performance will, in general, be small.

This conclusion is further strengthened by the following observations:

1) The calculated deflections and the effects of deflections are independent of the Young's modulus. Therefore, the decrease of Young's modulus at high temperature of the material will not influence the results of calculation.

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* See Appendix II

2) At higher temperatures, the Young's modulus will be decreased. Then with a given aerodynamic loading, the deflections due to this aerodynamic load will be increased at higher temperatures. Therefore, at the high temperature caused by aerodynamic heating, the deflections due to the aerodynamic load will be much larger than the deflections due to non-uniform heating.

3) If the aircraft is flown at a much higher altitude, the lower air density will greatly reduce the rate of heat input to the wing. Then the non-uniformity of temperature distribution over the material of the wing will be greatly reduced. This will in turn greatly decrease the importance of the deflections due to aerodynamic heating.

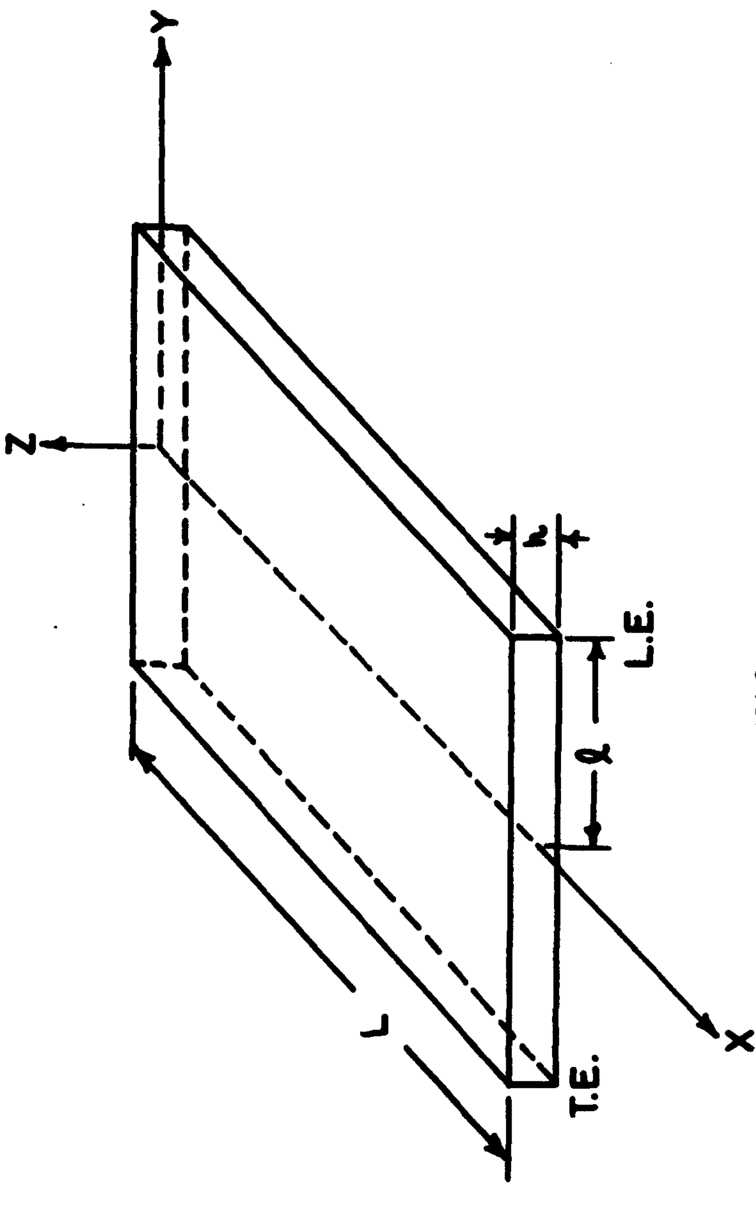



FIGURE 1

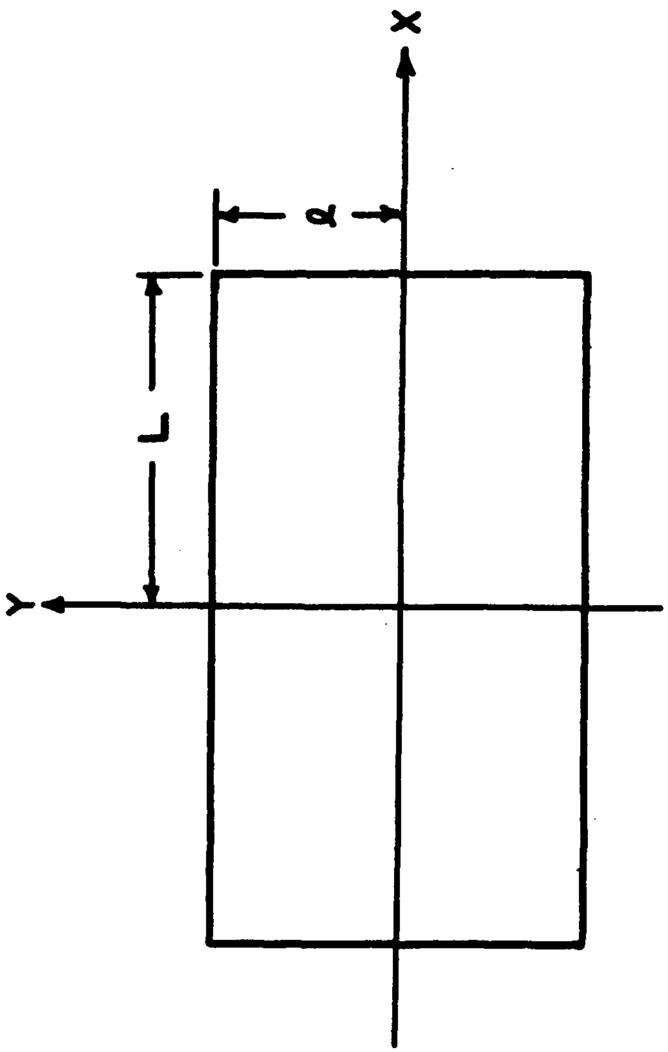



FIGURE 2

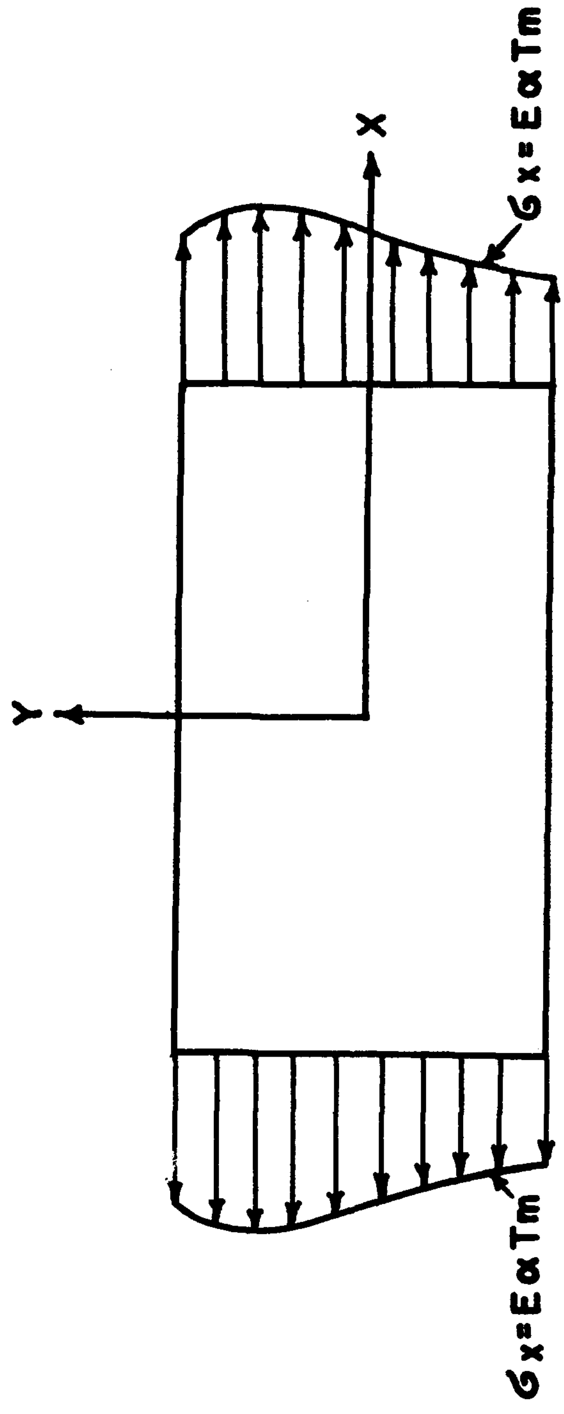



FIGURE 3

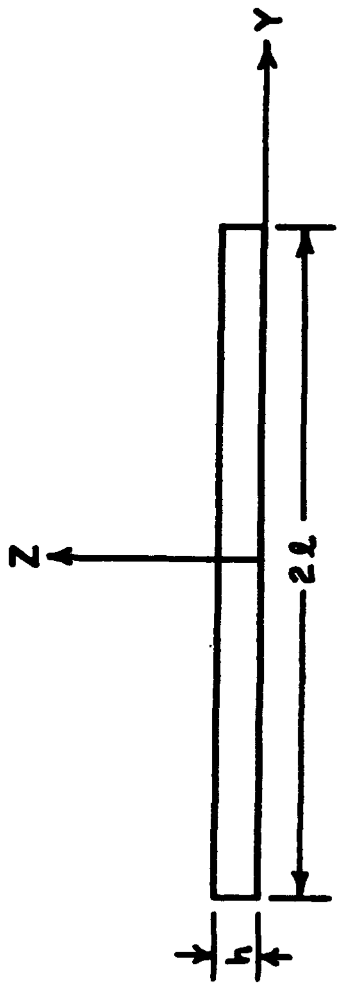
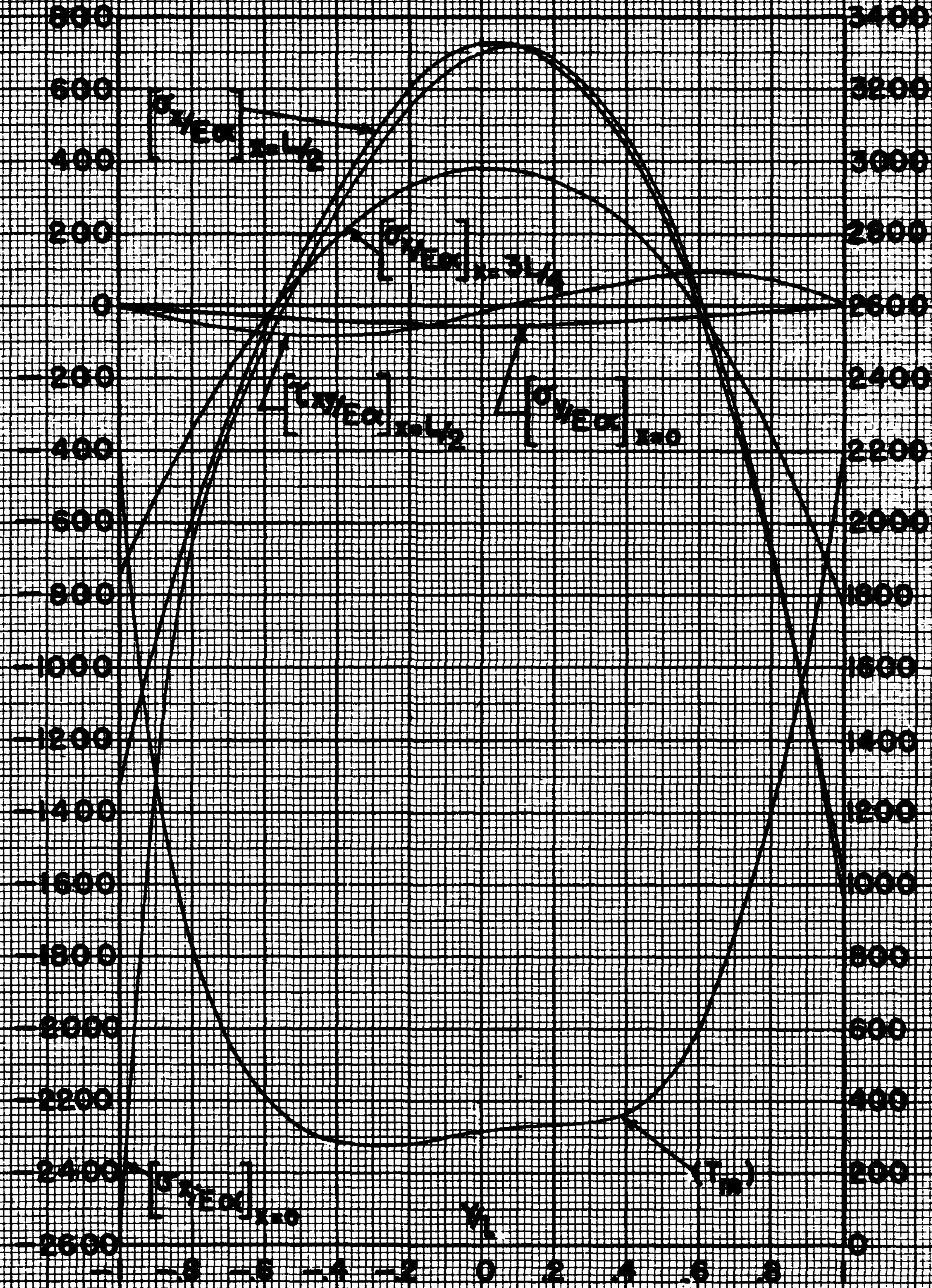


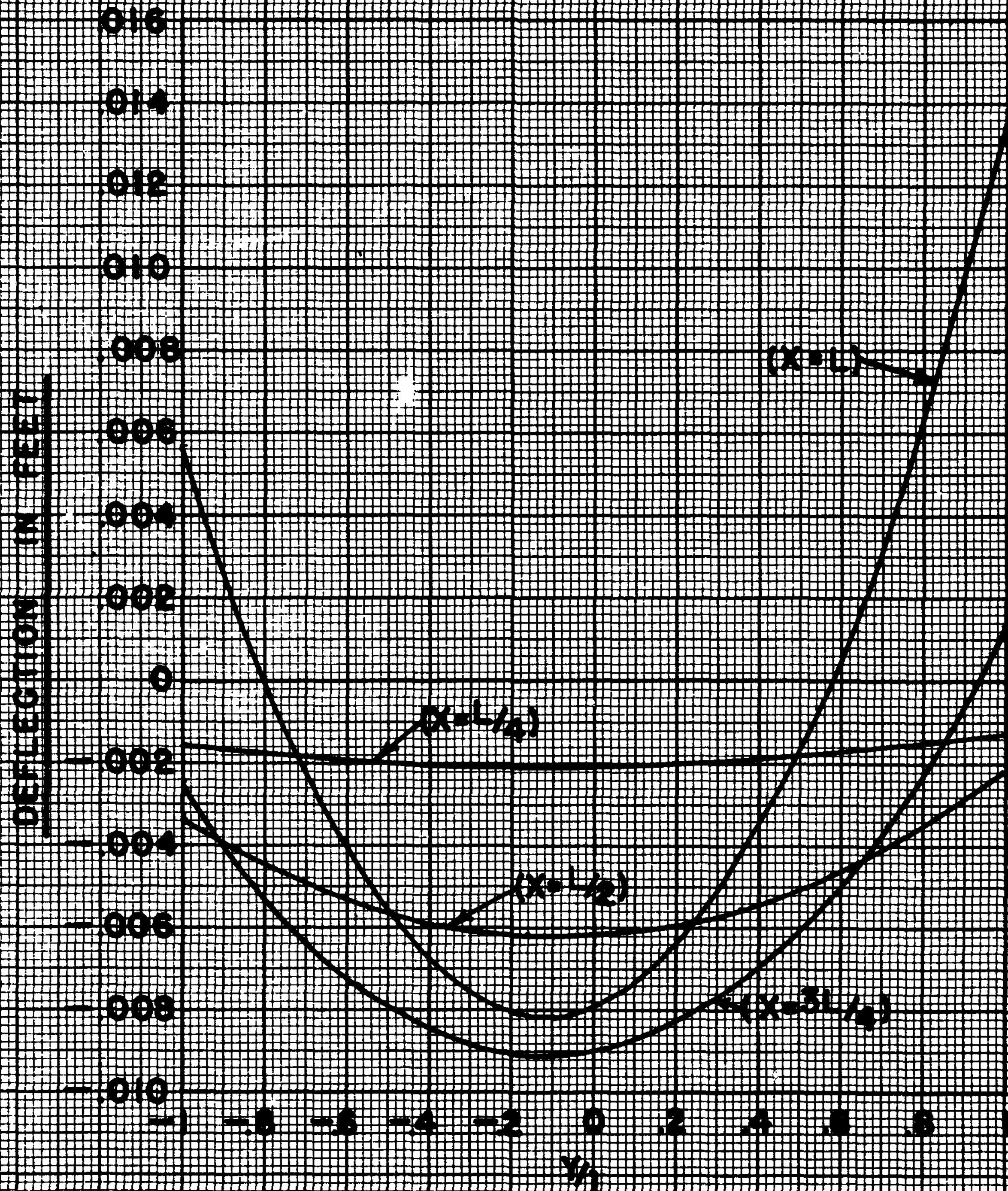

FIGURE 4

FIGURE 5



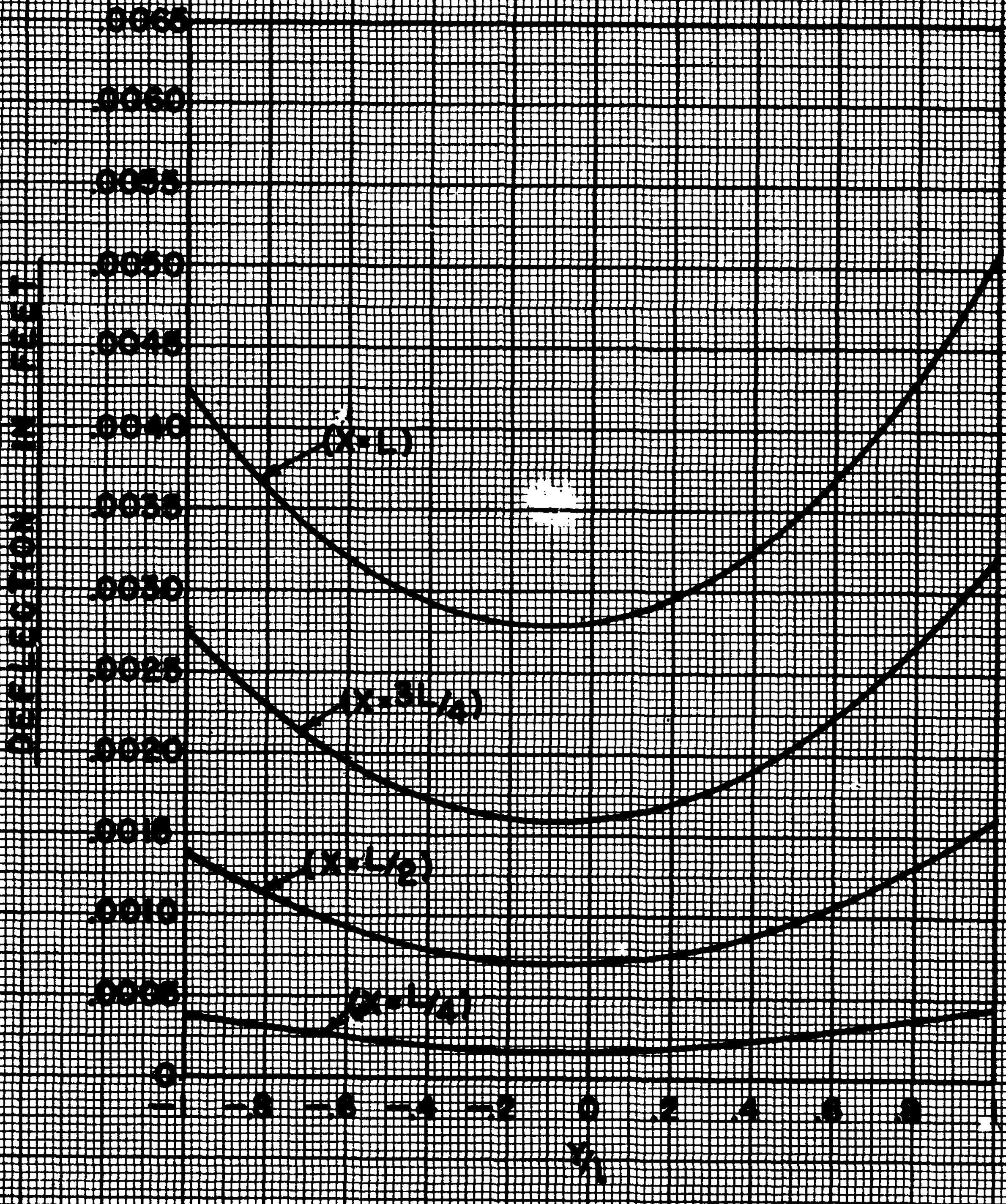

PLANE STRESS VS. CHORDWISE DISTANCE

FIGURE 6



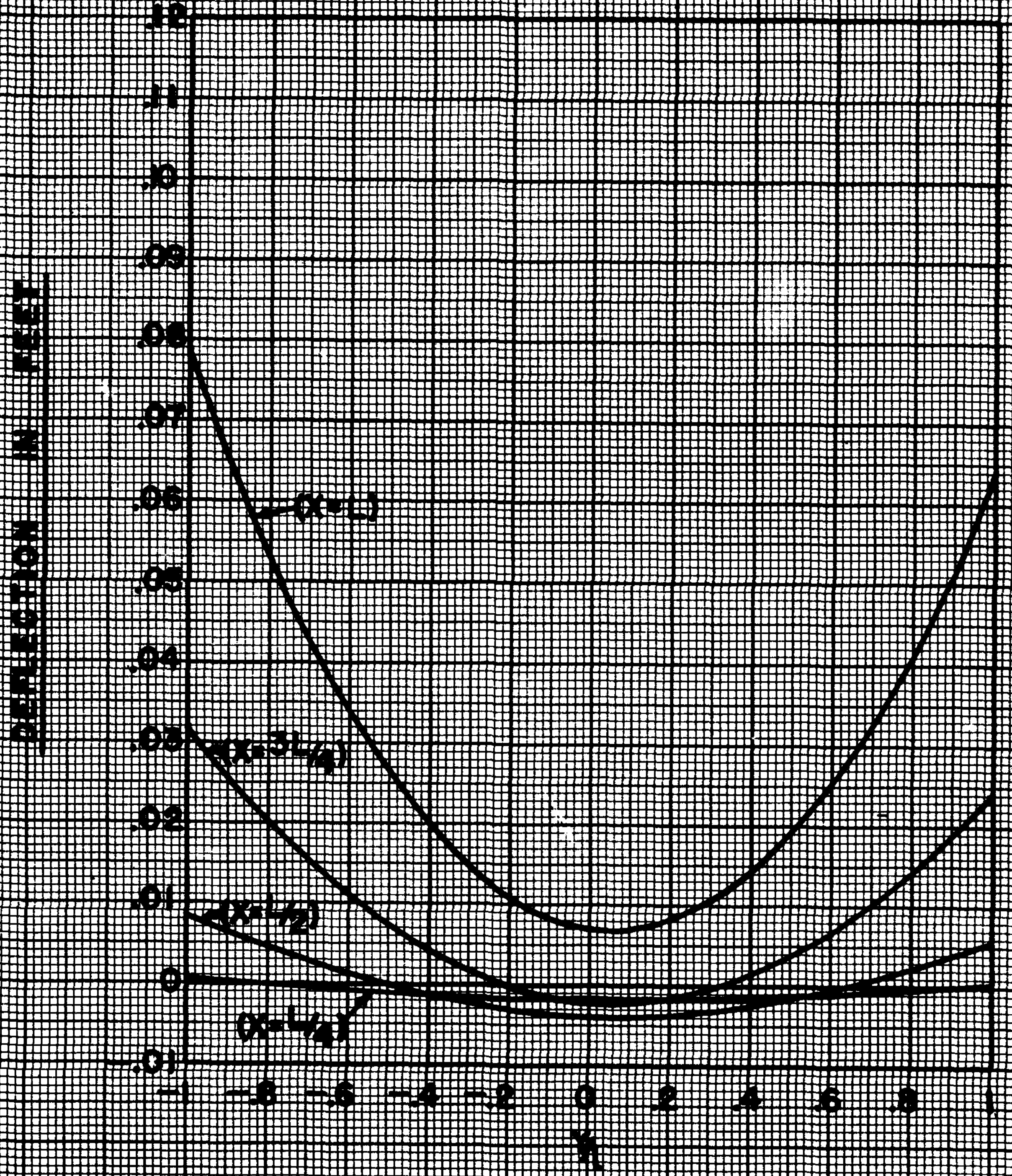

DEFLECTION VERSUS DISTANCE ALONG CHORD FOR R=3 AND $\alpha = 6.28 \times 10^{-6}/°F$

FIGURE 7



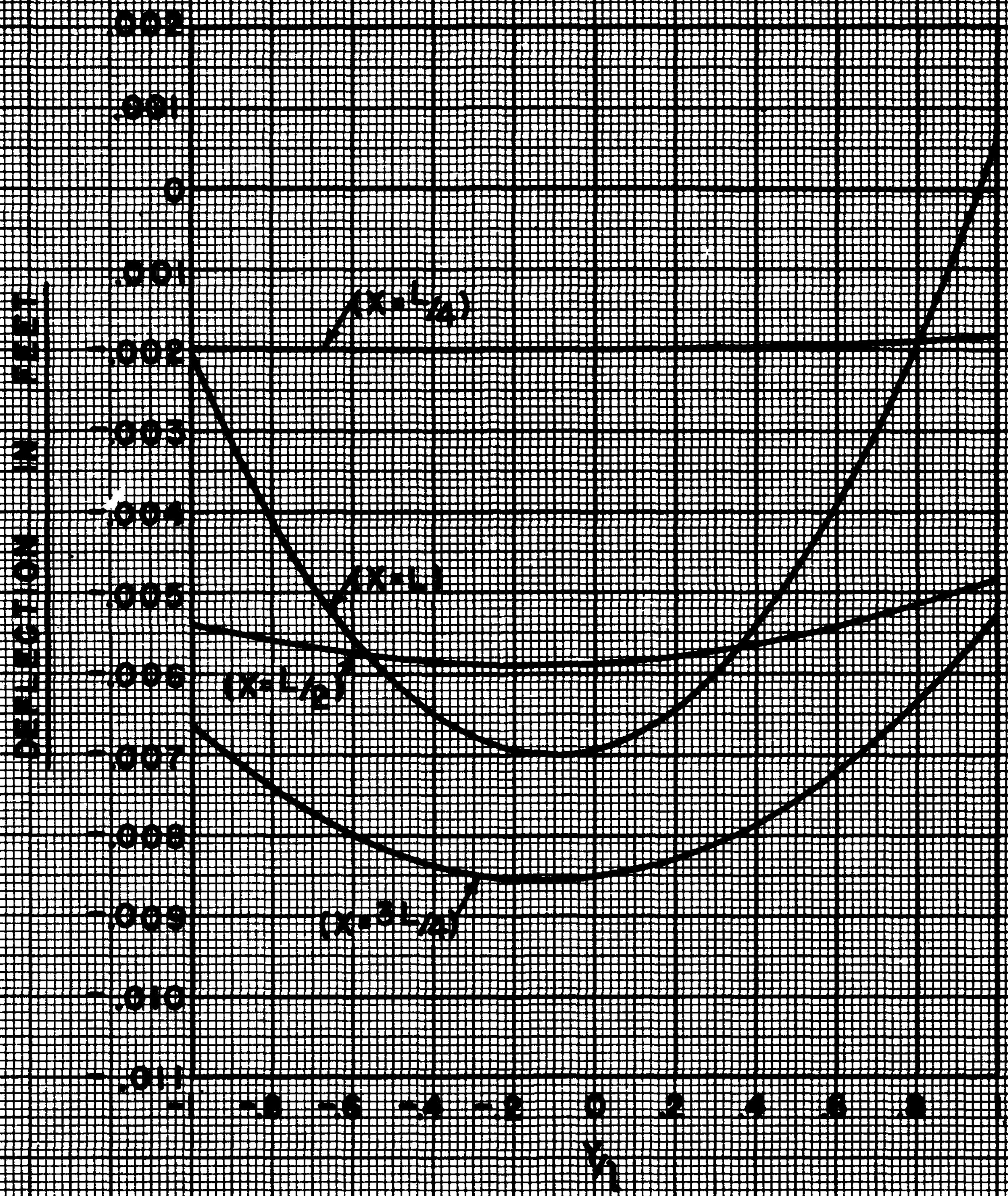

DEFLECTION VERSUS DISTANCE ALONG CHORD FOR $R = 1/2$ AND $\alpha = 6.28 \times 10^{-6}/{}^{\circ}F$

FIGURE 8



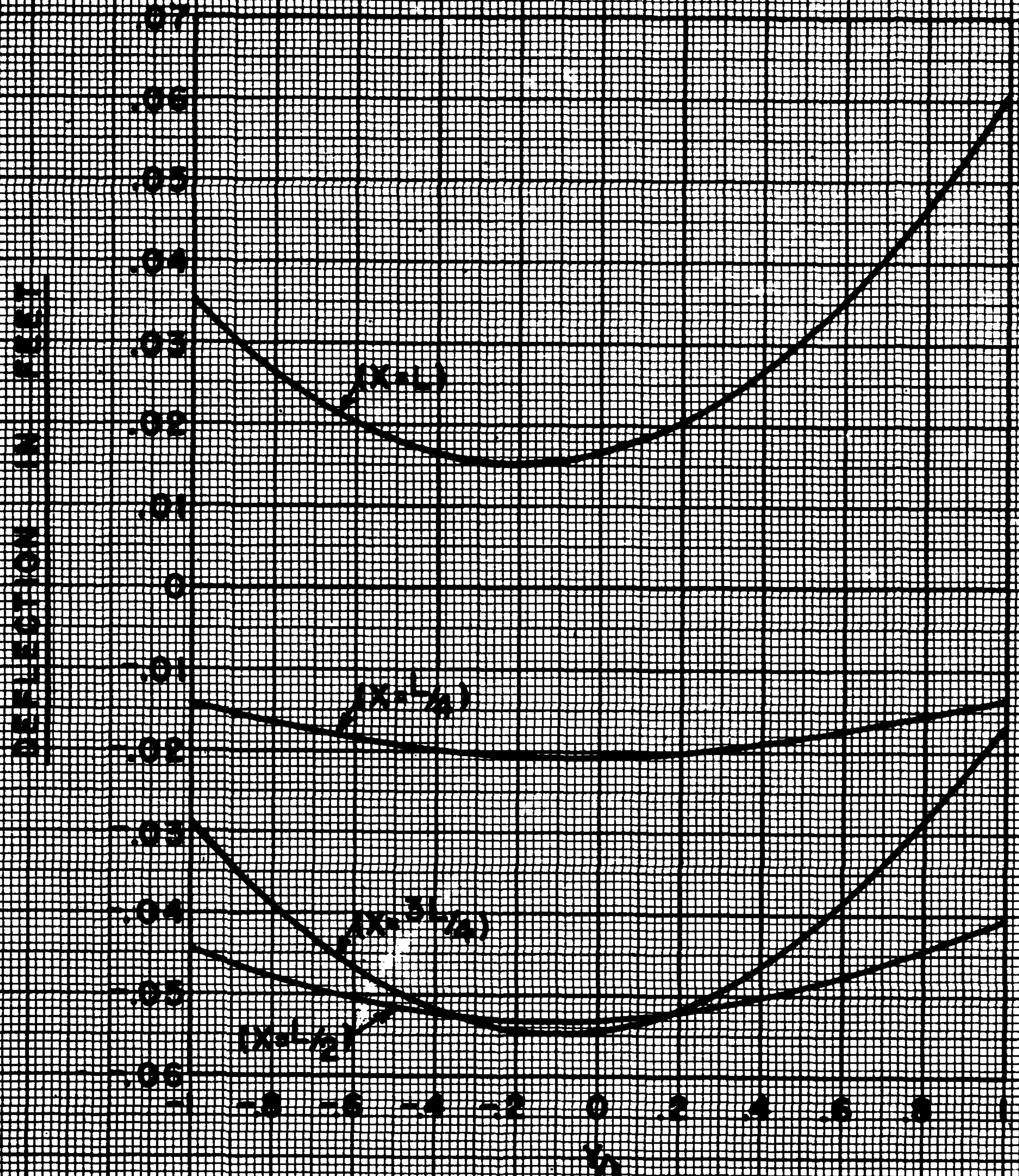

DEFLECTION VERSUS DISTANCE ALONG CHORD FOR R=3 AND $\alpha = 12.56 \times 10^{-6}/°F$

FIGURE 9




DEFLECTION VERSUS DISTANCE ALONG CHORD FOR R=3 AND $\alpha = 3.14 \times 10^{-6}/{}^{\circ}F$

**FIGURE 1D**




**DEFLECTION VERSUS DISTANCE ALONG CHORD FOR R=8 AND $\alpha = 6.28 \times 10^{-6}/{}^{\circ}F$**

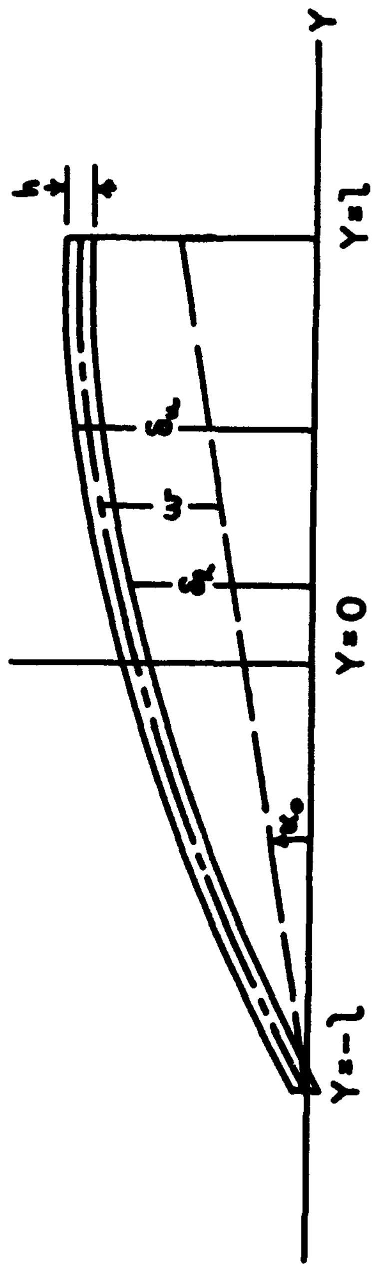



FIGURE 11

# TABLE I

| $R$ | 1/2 | 3 | 3 | 3 | 8 |
|---|---|---|---|---|---|
| $\alpha$ | $6.28 \times 10^{-6}/°F$ | $3.14 \times 10^{-6}/°F$ | $6.28 \times 10^{-6}/°F$ | $12.56 \times 10^{-6}/°F$ | $6.28 \times 10^{-6}/°F$ |
| $A_1$ | -43.4 | -19.5 | -19.5 | -19.5 | -2.415 |
| $A_2$ | -109 | -65.4 | -65.4 | -65.4 | -14.96 |
| $A_3$ | -23.7 | -3.22 | -3.22 | -3.22 | -0.469 |
| $a_1$ | .00296 | -.0399 | -.04053 | -.0342 | -.440 |
| $b_1$ | -.000145 | .0330 | .0328 | .0143 | .457 |
| $a_2$ | .000368 | .000848 | .00118 | -.00374 | .004525 |
| $b_2$ | .0000666 | .000454 | .002738 | -.00388 | .00795 |
| $a_3$ | .00446 | -.000253 | .00885 | .0191 | .1327 |
| $b_3$ | -.00262 | .00638 | .00853 | .0446 | -.102 |
| $C_L$ | .0167 | .01645 | .01661 | .0161 | .0172 |
| $C_M$ | .0000663 | .0001238 | .000416 | .001435 | .001529 |
| $C_D$ | .000398 | .000403 | .000415 | .000447 | .000500 |
| $\delta C_L / C_{L0}$ | .205% | .595% | 1.62% | -3.33% | 5.25% |
| $\delta C_D / C_{D0}$ | .454% | 1.67% | 4.84% | 12.81% | 26.2% |

## APPENDIX I

## TEMPERATURE DISTRIBUTION IN AN AERODYNAMICALLY HEATED WING

A wing is assumed to accelerate at a constant rate starting from steady flight at some supersonic Mach number. The adiabatic wall temperature which acts on the surface of the wing is thus a function of time, and a non-steady state heat transfer problem arises. It is assumed that there is no heat flow in the spanwise direction of the wing, and thus the problem is a two dimensional one. The coordinate system used on the section is as shown in Figure 4.

$$\tau = \frac{h}{2l}, \quad \zeta = \frac{z}{h}, \quad \eta = y/l$$

The differential equation of heat transfer for this system is:

$$\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \theta}$$

$$\frac{1}{h^2}\frac{\partial^2 T}{\partial \zeta^2} + \frac{1}{l^2}\frac{\partial^2 T}{\partial \eta^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \theta}$$

$$\text{or,} \quad \frac{\partial^2 T}{\partial \zeta^2} + 4\tau^2\frac{\partial^2 T}{\partial \eta^2} = \frac{h^2}{\alpha}\frac{\partial T}{\partial \theta}$$

Since $\partial^2 T/\partial \zeta^2$ and $\partial^2 T/\partial \eta^2$ are of the same order of magnitude, for small values of $\tau$ :

$$4\tau^2\frac{\partial^2 T}{\partial \eta^2} \ll \frac{\partial^2 T}{\partial \zeta^2}$$

Hence, assuming that $4\tau^2 \, \partial^2 T/\partial \eta^2$ is negligible, the differential equation becomes:

$$\frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \theta}$$

which is the one-dimensional equation.
Boundary conditions: The plate is assumed to be insulated at the surface $z = 0$, and heated according to Newton's law at the surface $z = h$. At time $\theta = 0$, the temperature is assumed to be identically zero. Thus the boundary conditions are:

$$\theta = 0 \quad : \quad T = 0$$

$$z = 0 \quad : \quad \partial T / \partial z = 0$$

$$z = h \quad : \quad K \frac{\partial T}{\partial z} = h_s \left( T_{aw} - T \right)$$

It is believed on the basis of experimental data that the surface coefficient of heat transfer, and the adiabatic wall temperature can be approximated by expressions of the form:

$$h_s / K = a + b\theta$$

$$T_{aw} = c\theta + d\theta^2$$

where a, b, c, and d are constants

With these two expressions, the boundary conditions become:

$$\theta = 0 \quad : \quad T = 0$$

$$z = 0 \quad : \quad \partial T / \partial z = 0$$

$$z = h \quad : \quad \partial T / \partial z = (a + b\theta)(c\theta + d\theta^2 - T)$$

The Laplace transform operator is now applied to both the differential equation and the boundary conditions. The operator is defined as follows:

$$\mathcal{L}\{f(\theta, z)\} = \int_0^\infty e^{-p\theta} f(\theta, z)\, d\theta = \bar{f}(p, z)$$

The result of this operation is:

$$\frac{d^2\overline{T}}{dZ^2} - \frac{p}{\alpha}\overline{T} = T_{\theta=0} = 0$$

$$Z = 0 \; ; \; \frac{\partial\overline{T}}{\partial Z} = 0$$

$$Z = h \; ; \; \frac{\partial\overline{T}}{\partial Z} = \frac{ac}{p^2} + \frac{2(ad+bc)}{p^3} + \frac{6bd}{p^4}$$

$$- a\overline{T} + b\frac{\partial\overline{T}}{\partial p}$$

The equation has thus been transformed to a second order total differential equation whose most general solution is:

$$\overline{T} = A(p)\cosh q Z + B(p)\sinh q Z$$

where : $q = \sqrt{p/\alpha}$

The first of the two boundary conditions determines that B = 0, and the second yields the following first order differential equation:

$$\frac{dA}{dp} + \left[\frac{h}{2\alpha q}\tanh qh - \frac{a}{b} - \frac{q}{b}\tanh qh\right]A$$

$$= -\frac{ac}{bp^2\cosh qh} - \frac{2(ad+bc)}{bp^3\cosh qh} - \frac{6d}{p^4\cosh qh}$$

Or, writing everything in terms of $q$:

$$\frac{dA}{dq} + \left[h \tanh qh - \frac{2a\alpha q}{b} - \frac{2\alpha q^2}{b} \tanh qh\right] A$$

$$= -\frac{1}{\cosh qh}\left[\frac{2ac}{b\alpha q^3} + \frac{4(ad+bc)}{b\alpha^2 q^5} + \frac{12d}{\alpha^3 q^7}\right]$$

But, $\frac{1}{\cosh qh} = 2\sum_{m=0}^{\infty}(-1)^m e^{-(2m+1)qh}$

Now, let $A(q) = \sum_{n=0}^{\infty}(-1)^n B_n(q)\, e^{-(2n+1)qh}$

and the differential equation becomes:

$$\sum_{n=0}^{\infty}(-1)^n e^{-(2n+1)qh}\left\{\frac{dB_n}{dq} - (2n+1)hB_n + \left[h\tanh qh - \frac{2\alpha a}{b} q - \frac{2\alpha}{b} q^2 \tanh qh\right] B_n\right\}$$

$$= -\sum_{m=0}^{\infty}(-1)^m e^{-(2m+1)qh}\left\{\frac{4ac}{b\alpha q^3} + \frac{8(ad+bc)}{b\alpha^2 q^5} + \frac{24d}{\alpha^3 q^7}\right\}$$

or, equating coefficients of $(-1)^n e^{-(2n+1)qh}$

$$\frac{dB_n}{dq} + \left[h\tanh qh - \frac{2\alpha a}{b} q - \frac{2\alpha q^2}{b}\tanh qh - (2n+1)h\right] B_n$$

$$= -\left[\frac{4ac}{b\alpha q^3} + \frac{8(ad+bc)}{b\alpha^2 q^5} + \frac{24d}{\alpha^3 q^7}\right]$$

The homogeneous solution to this equation can be obtained directly in the form of an integral:

$$B_{nH} = K e^{\int\left\{\frac{2\alpha a}{b} q + (2n+1)h + \frac{2\alpha q^2}{b}\tanh qh - h\tanh qh\right\} dq}$$

$$= K e^{\left\{\frac{\alpha a}{b} q^2 + (2n+1)hq + \int\left[\frac{2\alpha q^2}{b} - h\right]\tanh qh \, dq\right\}}$$

$$= K e^{J}$$

As p approaches infinity, the Laplace transform of the temperature must approach zero. This can be seen by the following analysis:

$$\bar{T} = \lim_{L\to\infty} \int_0^L e^{-p\theta}\, T(\theta)\, d\theta$$

$$= \int_0^{1/\sqrt{p}} e^{-p\theta}\, T\, d\theta + \lim_{L\to\infty} \int_{1/\sqrt{p}}^{L} e^{-p\theta}\, T\, d\theta$$

$$= M + N$$

The temperature T is bounded by the adiabatic wall temperature at all times. Hence:

$$T \leq c\theta + d\theta^2$$

and for $0 \leq \theta \leq 1/\sqrt{p}$ : $e^{-p\theta} \leq 1$ , $T \leq \frac{c}{\sqrt{p}} + \frac{d}{p}$

$$\frac{1}{\sqrt{p}} \leq \theta \leq L \; : \; e^{-p\theta} \leq e^{-\sqrt{p}}, \; T \leq cL + dL^2$$

Hence, by the theorem on upper bounds of integrals:

$$M \leq \left(\frac{c}{\sqrt{p}} + \frac{d}{p}\right)\left(\frac{1}{\sqrt{p}}\right) = \frac{c}{p} + \frac{d}{p^{3/2}}$$

$$N \leq \lim_{L \to \infty} \left[\left(L - \frac{1}{\sqrt{p}}\right)\left(cL + dL^2\right) e^{-\sqrt{p}}\right]$$

If p is now allowed to approach infinity, the products $Le^{-\sqrt{p}}$, $L^2e^{-\sqrt{p}}$, and $L^3 e^{-\sqrt{p}}$ approach zero, and $1/p$ , $1/p^{3/2}$ approach zero. Hence both M and N approach zero, and:

$$\lim_{p \to \infty} T = \lim_{p \to \infty} A \cosh q_b z = 0$$

The homogeneous solution of $B_n$ however contains a factor of $e^{q^2}$ This factor produces a contribution to A of the form:

$$A_H = \sum_{n=0}^{\infty} (-1)^n f(q) e^{\left[q^2 - (2n+1) h q\right]}$$

$$= \sum_{n=0}^{\infty} (-1)^n f(q) e^{q\left[q - (2n+1) h\right]}$$

where $f(q)$ does not approach zero when P becomes infinite. But this contribution becomes infinite when P becomes infinite, and hence contradicts the requirement that A approach zero. Thus, the coefficient K of the homogeneous solution must be zero. The particular solution is found by the method of variation of parameters. A solution is assumed in the form.

$$B_{np} = f(q) e^{J}$$

When introduced in the differential equation, the function f is expressed as a quadrature.

$$\frac{df}{dq} = -e^{-J}\left[\frac{4ac}{b\alpha q^3} + \frac{8(ad+bc)}{b\alpha^2 q^5} + \frac{24d}{\alpha^3 q^7}\right]$$

or: $$f = -\int e^{-J}\left[\frac{4ac}{b\alpha q^2} + \frac{8(ad+bc)}{b\alpha^2 q^5} + \frac{24d}{\alpha^3 q^7}\right] dq$$

If this expression is integrated by parts repeatedly, an asymptotic expansion for f is determined, which contains increasing negative powers of $q$. This expansion is as follows:

$$f = e^{-J}\left\{\frac{\left[\frac{4ac}{b\alpha q^3} + \frac{8(ad+bc)}{b\alpha^2 q^5} + \frac{24d}{\alpha^3 q^7}\right]}{dJ/dq} + \frac{1}{dJ/dq}\frac{d}{dq}\left[\frac{\left(\frac{4ac}{b\alpha q^3} + \frac{8(ad+bc)}{b\alpha^2 q^5} + \frac{24d}{\alpha^3 q^7}\right)}{dJ/dq}\right]\right.$$

$$\left. + \frac{1}{dJ/dq}\frac{d}{dq}\left\langle\frac{1}{dJ/dq}\frac{d}{dq}\left[\frac{\left(\frac{4ac}{b\alpha q^3} + \frac{8(ad+bc)}{b\alpha^2 q^5} + \frac{24d}{\alpha^3 q^7}\right)}{dJ/dq}\right]\right\rangle + \cdots\right\}$$

$$-\int e^{-J}\left\{\frac{1}{dJ/dq}\frac{d}{dq}\left[\frac{1}{dJ/dq}\frac{d}{dq}\left\langle \cdots \left(\frac{\frac{4ac}{b\alpha q^3} + \frac{8(ad+bc)}{b\alpha^2 q^5} + \frac{24d}{\alpha^3 q^7}}{dJ/dq}\right)\right\rangle\right]\right\} dq$$

$$\frac{dJ}{dq} = \left(2\frac{\alpha q^2}{b} - h\right)\tanh qh + \frac{2\alpha a}{b}q + (2n+1)h$$

But: $$\tanh qh = 1 + 2\sum_{m=1}^{\infty}(-1)^m e^{-2mqh}$$

Hence: $\frac{dJ}{dq_b} = \frac{2\alpha q_b^2}{b} + \frac{2\alpha a}{b} q_b + 2nh + \left(\frac{4\alpha q_b^2}{b} - 2h\right)\sum_{m=1}^{\infty}(-1)^m e^{-2mq_bh}$

The factor $1/\frac{dJ}{dq_b}$ can now be expanded in inverse powers of $q_b$, to yield:

$$
\begin{aligned}
1/\frac{dJ}{dq_b} = {} & \frac{1}{q_b^2}\left\{\frac{b}{2\alpha}\left[1 + 2e^{-2q_bh} + 2e^{-4q_bh} + 2e^{-6q_bh} + \cdots\right]\right\} \\
& - \frac{1}{q_b^3}\left\{\frac{ab}{2\alpha}\left[1 + 4e^{-2q_bh} + 8e^{-4q_bh} + 12e^{-6q_bh} + \cdots\right]\right\} \\
& + \frac{1}{q_b^4}\left\{\frac{a^2b}{2\alpha}\left[1 + 6e^{-2q_bh} + 18e^{-4q_bh} + 38e^{-6q_bh} + \cdots\right]\right. \\
& \qquad \left. - \frac{hb^2}{2\alpha^2}\left[n + (4n+1)e^{-2q_bh} + (8n+3)e^{-4q_bh} + (12n+5)e^{-6q_bh} + \cdots\right]\right\} \\
& - \frac{1}{q_b^5}\left\{\frac{a^3b}{2\alpha}\left[1 + 8e^{-2q_bh} + 32e^{-4q_bh} + 88e^{-6q_bh} + \cdots\right]\right. \\
& \qquad \left. - h\frac{ab^2}{\alpha^2}\left[n + (6n+1)e^{-2q_bh} + (18n+5)e^{-4q_bh} + (38n+13)e^{-6q_bh} + \cdots\right]\right\} \\
& + \frac{1}{q_b^6}\left\{\frac{a^4b}{2\alpha}\left[1 + 10e^{-2q_bh} + 50e^{-4q_bh} + 170e^{-6q_bh} + \cdots\right]\right. \\
& \qquad - \frac{3ha^2b^2}{2\alpha^2}\left[n + (8n+1)e^{-2q_bh} + (27n+7)e^{-4q_bh} + (88n+25)e^{-6q_bh} + \cdots\right] \\
& \qquad + \frac{h^2b^3}{2\alpha^3}\left[n^2 + 2n(3n+1)e^{-2q_bh} + (18n^2+10n+1)e^{-4q_bh}\right. \\
& \qquad \left.\left. + 2(19n^2+13n+2)e^{-6q_bh} + \cdots\right]\right\} + \cdots
\end{aligned}
$$

Introducing this into the expression for $f(q)$, and arranging terms in order of negative powers of $q$, the expression becomes:

$$B_n = B_{nH} + B_{np} = B_{np} = f(q)\,e^{J}$$

$$= \frac{1}{q^5}\left\{\frac{2ac}{\alpha^2}\left[1 + 2e^{-2qh} + 2e^{-4qh} + 2e^{-6qh} + \cdots\right]\right\}$$

$$-\frac{1}{q^6}\left\{\frac{2a^2c}{\alpha^2}\left[1 + 4e^{-2qh} + 8e^{-4qh} + 12e^{-6qh} + \cdots\right]\right\}$$

$$+\frac{1}{q^7}\left\{\frac{2a^3c}{\alpha^2}\left[1 + 6e^{-2qh} + 18e^{-4qh} + 38e^{-6qh} + \cdots\right]\right.$$

$$+\frac{4(ad+bc)}{\alpha^3}\left[1 + 2e^{-2qh} + 2e^{-4qh} + 2e^{-6qh} + \cdots\right]$$

$$\left.-\frac{2abch}{\alpha^3}\left[n + (4n+3)e^{-2qh} + (8n+11)e^{-4qh} + (12n+25)e^{-6qh} + \cdots\right]\right\}$$

$$-\frac{1}{q^8}\left\{\frac{2a^4c}{\alpha^2}\left[1 + 8e^{-2qh} + 32e^{-4qh} + 88e^{-6qh} + \cdots\right]\right.$$

$$+\frac{a(9bc+4ad)}{\alpha^3}\left[1 + 4e^{-2qh} + 8e^{-4qh} + 12e^{-6qh} + \cdots\right]$$

$$\left.-\frac{4a^2bch}{\alpha^3}\left[n + (6n+4)e^{-2qh} + (18n+28)e^{-4qh} + (38n+73)e^{-6qh} + \cdots\right]\right\}$$

$$+\frac{1}{q^9}\left\{\frac{2a^5c}{\alpha^2}\left[1 + 10e^{-2qh} + 50e^{-4qh} + 170e^{-6qh} + \cdots\right]\right.$$

$$+\frac{12bd}{\alpha^4}\left[1 + 2e^{-2qh} + 2e^{-4qh} + 2e^{-6qh} + \cdots\right]$$

$$-\frac{4(ad+bc)bh}{\alpha^4}\left[n + (4n+3)e^{-2qh} + (8n+11)e^{-4qh} + (12n+25)e^{-6qh} + \cdots\right]$$

$$-\frac{2a^3bch}{\alpha^3}\left[3n+(24n+16)e^{-2qh}+(81n+118)e^{-4qh}+(264n+488)e^{-6qh}+\dots\right]$$

$$+\frac{2acb^2h^2}{\alpha^4}\left[n^2+(6n^2+8n)e^{-2qh}+(18n^2+46n+7)e^{-4qh}+(38n^2+146n+34)e^{-6qh}+\dots\right]$$

$$+\frac{a^2bc}{\alpha^3}\left[15+\left(90-\frac{2bh^2}{\alpha a}\right)e^{-2qh}+\left(270-\frac{8bh^2}{\alpha a}\right)e^{-4qh}+\left(570-\frac{18bh^2}{\alpha a}\right)e^{-6qh}+\dots\right]$$

$$+\frac{4a^3d}{\alpha^3}\left[1+6e^{-2qh}+18e^{-4qh}+38e^{-6qh}+\dots\right]\Bigg\}+\dots$$

But, $$T=\sum_{n=0}^{\infty}\frac{(-1)^n}{2}B_n\left[e^{-q\langle(2n+1)h-y\rangle}+e^{-q\langle(2n+1)h+y\rangle}\right]$$

and making use of the inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{e^{-qx}}{q^n}\right\}=\alpha^{n/2}\,\mathcal{L}^{-1}\left\{\frac{e^{-qx}}{p^{n/2}}\right\}$$

$$=\alpha^{n/2}(4\theta)^{\frac{n-1}{2}}\,i^{n-2}\mathrm{erfc}\,\frac{x}{2\sqrt{\alpha\theta}}$$

The following expression for the temperature is derived:

$$T=\sum_{m=0}^{\infty}\sum_{n=3}^{\infty}C_{mn}(4\theta)^{n/2}\left[i^n\mathrm{erfc}\,\mu(2m+1-\zeta)+i^n\mathrm{erfc}\,\mu(2m+1+\zeta)\right]$$

where $\mu=\dfrac{h}{2\sqrt{\alpha\theta}}$

$\zeta=z/h$

and where the coefficients $C_{mn}$, are:

$$C_{03} = ac\sqrt{\alpha} \qquad C_{04} = -a^2 c\alpha$$

$$C_{13} = ac\sqrt{\alpha} \qquad C_{14} = -3a^2 c\alpha$$

$$C_{23} = ac\sqrt{\alpha} \qquad C_{24} = -5a^2 c\alpha$$

$$C_{33} = ac\sqrt{\alpha} \qquad C_{34} = -7a^2 c\alpha$$

$$C_{05} = a^3 c\alpha^{3/2} + 2(ad+bc)\sqrt{\alpha}$$

$$C_{15} = 5a^3 c\alpha^{3/2} + 2(ad+bc)\sqrt{\alpha} - 2abch\sqrt{\alpha}$$

$$C_{25} = 13a^3 c\alpha^{3/2} + 2(ad+bc)\sqrt{\alpha} - 6abch\sqrt{\alpha}$$

$$C_{35} = 25a^3 c\alpha^{3/2} + 2(ad+bc)\sqrt{\alpha} - 14abch\sqrt{\alpha}$$

$$C_{06} = -a^4 c\alpha^2 - 2(ad+bc)a\alpha - \tfrac{5}{2}abc\alpha$$

$$C_{16} = -7a^4 c\alpha^2 - 6(ad+bc)a\alpha - \tfrac{15}{2}abc\alpha + 6a^2bch\alpha$$

$$C_{26} = -25a^4 c\alpha^2 - 10(ad+bc)a\alpha - \tfrac{25}{2}abc\alpha + 30a^2bch\alpha$$

$$C_{36} = -63a^4 c\alpha^2 - 30(ad+bc)a\alpha - \tfrac{75}{2}abc\alpha + 90a^2bch\alpha$$

$$C_{07} = a^5 c\alpha^{5/2} + 2(ad+bc)a^2\alpha^{3/2} + \tfrac{11}{2}a^2bc\alpha^{3/2} + 6bd\sqrt{\alpha}$$

$$C_{17} = 8a^5 c\alpha^{5/2} + 10(ad+bc)a^2\alpha^{3/2} + \tfrac{55}{2}a^2bc\alpha^{3/2} + 6bd\sqrt{\alpha}$$
$$-4(ad+bc)bh\sqrt{\alpha} - 12a^3bch\alpha^{3/2} - 2acb^2h^2\sqrt{\alpha}$$

$$C_{27} = 41a^5 c\alpha^{5/2} + 26(ad+bc)a^2\alpha^{3/2} + \tfrac{143}{2}a^2bc\alpha^{3/2} + 8bd\sqrt{\alpha}$$
$$-12(ad+bc)bh\sqrt{\alpha} - 80a^3bch\alpha^{3/2} - 6acb^2h^2\sqrt{\alpha}$$

$$C_{37} = 129a^5 c\alpha^{5/2} + 50(ad+bc)a^2\alpha^{3/2} + \tfrac{275}{2}a^2bc\alpha^{3/2} + 6bd\sqrt{\alpha}$$
$$-28(ad+bc)bh\sqrt{\alpha} - 343a^3bch\alpha^{3/2} - 12acb^2h^2\sqrt{\alpha}$$

$$C_{08} = -a^6 c\alpha^3 - 2(ad+bc)a^3\alpha^2 - \tfrac{19}{2}a^3bc\alpha^2 - 6abd\alpha - 7(ad+bc)b\alpha$$

$$C_{09} = a^7 c\alpha^{7/2} + 2(ad+bc)a^4\alpha^{5/2} + \tfrac{26}{2}a^4bc\alpha^{5/2} + 6a^2bd\alpha^{3/2}$$
$$+15(ad+bc)ab\alpha^{3/2} + 10a^2bc\alpha^{3/2}$$

$$C_{0,10} = -a^8 c\alpha^4 - 2(ad+bc)a^5\alpha^3 - {}^{35}/_{2}\, a^5 bc\alpha^3 - 6\, a^3 bd\, \alpha^2$$
$$- 24(ad+bc)a^2 b\alpha^2 - {}^{189}/_{4}\, a^2 b^2 c\,\alpha^2 - 27\, b^2 d\,\alpha$$

$$C_{0,11} = a^9 c\alpha^{9/2} + 2(ad+bc)a^6\alpha^{7/2} + {}^{45}/_{2}\, a^6 bc\alpha^{7/2} + 6\, a^4 bd\,\alpha^{5/2}$$
$$+ 34(ad+bc)a^3 b\alpha^{5/2} + {}^{319}/_{4}\, a^3 b^2 c\alpha^{5/2} + 57\, ab^2 d\,\alpha^{3/2}$$
$$+ 35(ad+bc)b^2\alpha^{3/2}$$

$$C_{0,12} = -a^{10} c\alpha^5 - 2(ad+bc)\,a^7\alpha^4 - {}^{57}/_{2}\, a^7 bc\,\alpha^4 - 6\, a^5 bd\,\alpha^3$$
$$- 40(ad+bc)a^4 b\alpha^3 - {}^{605}/_{4}\, a^4 b^2 c\,\alpha^3 - 75\, a^2 b^2 d\alpha^2$$
$$- {}^{285}/_{2}(ad+bc)ab^2\alpha^2 - 45\, ab^3 c\alpha^2$$

As a check on this result, numerical calculations have been made for two points, and compared with results of a relaxation procedure. The series was found to converge very rapidly with respect to m, but not very rapidly with respect to n. The convergence is improved greatly at smaller values of $\theta$. The calculations were made for the following set of conditions;

Acceleration = 1 Mach number per 30 seconds starting from $M_0 = 1.4$

$a = .909$ 1/ft

$b = 136.3$ 1/ft -hr

$c = 26{,}800$ °F/hr

$d = 776{,}000$ °F/hr$^2$

$h = 0.20$ ft

$\alpha = .384$ ft$^2$/hr

$\zeta = 1$

For $\theta = .0383$ hrs, or $M = 6$:

$$\Delta T = T_{\theta} - T_{\theta=0} = 660\ {}^{\circ}F$$

For $\theta$ = .02165 hrs, or M = 4:

$$\Delta T = 160.5\ ^\circ F$$

The relaxation procedure gives values of 672 $^\circ$F, and 168.5$^\circ$F respectively for these conditions.

| M | $\Delta T$ calculated | $\Delta T$ relaxation | % difference |
|---|---|---|---|
| 4 | 160.5$^\circ$F | 168.5$^\circ$F | 4.75 |
| 6 | 660$^\circ$F | 672$^\circ$F | 1.79 |

These percentage differences are within the error of the finite difference procedure, so that this should serve as a valid check on the analytical procedure.

## APPENDIX II

### EFFECT OF DEFLECTIONS ON AERODYNAMIC COEFFICIENTS

In this appendix, the changes in aerodynamic coefficients due to the thermal deflections are calculated. The work is based upon the Ackeret thin wing approximations, i.e., the flow around every chordwise section of the wing will be considered as two-dimensional and the effect of the tip in reducing the fluid pressure will be neglected. For the configuration shown in figure 11, the Ackeret formulae for pressure coefficients on the upper and lower surfaces of the wing are as follows:

$$C_p = \frac{p - p^\circ}{\frac{1}{2}\rho^\circ U^2}$$

$$C_{p\ell} = \frac{2}{\sqrt{M^2 - 1}} \frac{d\delta_\ell}{dy}$$

$$C_{pu} = -\frac{2}{\sqrt{M^2 - 1}} \frac{d\delta_u}{dy}$$

$$\delta_u = \alpha_0(\ell + y) + w + h/2$$

$$\delta_\ell = \alpha_0(\ell + y) + w - h/2$$

But, $$w = A(\xi) + B(\xi)\eta + C(\xi)\eta^2$$

where, $$A(\xi) = a_1\xi^2 + b_1\xi^3$$

$$B(\xi) = a_2\xi^2 + b_2\xi^3$$

$$C(\xi) = a_3\xi^2 + b_3\xi^3$$

Hence:

$$\frac{d\delta_u}{dy} = \alpha_0 + B/\ell + 2C\,y/\ell^2$$

$$\frac{d\delta_\ell}{dy} = \alpha_0 + B/\ell + 2C\,y/\ell^2$$

$$C_{p\ell} - C_{pu} = \frac{2}{\sqrt{M^2-1}}\left(\frac{d\delta_\ell}{dy} + \frac{d\delta_u}{dy}\right)$$

$$= \frac{4}{\sqrt{M^2-1}}\left(\alpha_0 + B/\ell + 2C\,y/\ell^2\right)$$

**1. Lift coefficient**

$$C_\ell = \frac{\text{LIFT}}{1/2\,\rho^0\,U^2(2\ell)\cdot 1}$$

$$= \frac{1}{\rho^0 U^2 \ell}\int_{-\ell}^{\ell}(P_\ell - P_u)\,dy$$

$$= \frac{1}{2\ell}\int_{-\ell}^{\ell}(C_{p\ell} - C_{pu})\,dy$$

$$= \frac{2}{\ell\sqrt{M^2-1}}\int_{-\ell}^{\ell}\left(\alpha_0 + B/\ell + 2C\,y/\ell^2\right)dy$$

$$= \frac{4}{\sqrt{M^2-1}}\left(\alpha_0 + B/\ell\right)$$

2. Moment coefficient

$$C_m = \frac{\text{NOSE UP MOMENT ABOUT MIDCHORD}}{\frac{1}{2}\rho^0 U^2 (2\ell)^2 \cdot 1}$$

$$= \frac{1}{(2\ell)^2} \int_{-\ell}^{\ell} (C_{p\ell} - C_{pu})\, y \, dy$$

$$= \frac{1}{\ell^2\sqrt{M^2-1}} \int_{-\ell}^{\ell} \left(\alpha_0 y + B\, y/\ell + 2C\, y^2/\ell^2\right) dy$$

$$= \frac{1}{\ell^2\sqrt{M^2-1}} \left(4/3\, C\ell\right)$$

$$= \frac{4C}{3\ell\sqrt{M^2-1}}$$

3. Drag coefficient

$$C_d = \frac{\text{DRAG}}{\frac{1}{2}\rho^0 U^2 (2\ell)}$$

$$= \frac{1}{\rho^0 U^2 \ell} \int_{-\ell}^{\ell} \left(P_\ell \frac{d\delta_\ell}{dy} - P_u \frac{d\delta_u}{dy}\right) dy$$

$$= \frac{1}{2\ell} \int_{-\ell}^{\ell} (C_{p\ell} - C_{pu})\left(\alpha_0 + B/\ell + 2C\, y/\ell^2\right) dy$$

$$= \frac{2}{\ell\sqrt{M^2-1}} \int_{-\ell}^{\ell} \left(\alpha_0 + B/\ell + 2C\, y/\ell^2\right)^2 dy$$

$$= \frac{4}{\sqrt{M^2-1}} \left[\frac{B^2 + 4/3\, C^2}{\ell^2} + \alpha_0^2 + \frac{2\alpha_0 B}{\ell}\right]$$

4. "Average" lift coefficient

$$C_L = \int_0^1 C_l \, d\zeta$$

$$= \frac{4}{\sqrt{M^2-1}} \int_0^1 \left(\alpha_0 + a_2\zeta^2/l + b_2\zeta^3/l\right) d\zeta$$

$$= \frac{4}{\sqrt{M^2-1}} \left(\alpha_0 + a_2/3l + b_2/4l\right)$$

$$= \frac{1}{\sqrt{M^2-1}} \left(4\alpha_0 + \frac{4a_2 + 3b_2}{3l}\right)$$

5. "Average moment coefficient

$$C_M = \int_0^1 C_m \, d\zeta$$

$$= \frac{4}{3l\sqrt{M^2-1}} \int_0^1 \left(a_3\zeta^2 + b_3\zeta^3\right) d\zeta$$

$$= \frac{4}{3l\sqrt{M^2-1}} \left(a_3/3 + b_3/4\right)$$

$$= \frac{4a_3 + 3b_3}{9l\sqrt{M^2-1}}$$

6. "Average" drag coefficient

$$C_D = \int_0^1 C_d \, d\zeta$$

$$= \frac{4}{\sqrt{M^2-1}} \int_0^1 \Big\{ \frac{1}{l^2}\Big[a_2^2\zeta^4 + 2a_2b_2\zeta^5 + b_2^2\zeta^6 + 4/3\, a_3^2\zeta^4 + 8/3\, a_3b_3\zeta^5$$

$$+ 4/3\, b_3^2\zeta^6\Big] + \alpha_0^2 + \frac{2\alpha_0}{l}\left[a_2\zeta^2 + b_2\zeta^3\right]\Big\} d\zeta$$

$$= \frac{4}{\sqrt{M^2-1}} \Big\{ \frac{1}{l^2}\Big[a_2^2/5 + a_2b_2/3 + b_2^2/7 + 4a_3^2/15 + 4a_3b_3/9$$

$$+ 4b_3^2/21\Big] + \alpha_0^2 + \frac{\alpha_0}{l}\left[\frac{2a_2}{3} + \frac{b_2}{2}\right]\Big\}$$

For $W/S = 100\ \#/ft^2$

$\alpha_e = .0242$ radians

A sample calculation will now be made for the case of a wing of $R = 3$, and $\alpha = 6.28 \times 10^{-6}$. Results of calculations for other wings are tabulated in table 1.

$$a_2 = .00118$$

$$b_2 = .002738$$

$$a_3 = .00885$$

$$b_3 = .00853$$

Hence: $$C_L = \frac{1}{\sqrt{35}}\left\{4(.0242) + \frac{4(.00118) + 3(.002738)}{3(2.75)}\right\}$$

$$= \frac{1}{5.92}\left\{.0968 + .001568\right\}$$

$$= .01661$$

$$C_M = \frac{4(.00885) + 3(.00853)}{9(2.75)(5.92)}$$

$$= .000418$$

$$C_D = \frac{4}{5.92}\left\{\frac{1}{(2.75)^2}\left[\frac{(.00118)^2}{5} + \frac{(.00118)(.002738)}{3} + \frac{(.002738)^2}{7}\right.\right.$$

$$\left. + \frac{4(.00885)^2}{15} + \frac{4(.00885)(.00853)}{9} + \frac{4(.00853)^2}{21}\right]$$

$$\left. + (.0242)^2 + \frac{.0242}{2.75}\left[\frac{2(.00118)}{3} + \frac{.002738}{2}\right]\right\}$$

$$= .676\left\{.00000837 + .000586 + .0000190\right\}$$

$$= .000415$$

The percentage changes of lift and drag coefficients are defined as:

$$\frac{\text{change in coefficient due to deflection}}{\text{undeflected coefficient}} \times 100$$

$$\%\ \text{change in } C_L = \frac{.001568}{.0966} \times 100 = 1.62\ \%$$

$$\%\ \text{change in } C_D = \frac{.00000937 + .0000180}{.000566} \times 100$$

$$= 4.84\ \%$$

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